BUILDING CONTINUOUS TIME CRYSTALS FROM RARE EVENTS

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R. Hurtado
F. Carollo
C.P. Espigares

Sketch of the talk

1) Crash course on large deviations (LDs)

2) Large deviations in diffusive systems

3) Time crystal phase in WASEP

4) Making rare events typical

5) Time crystal lattice gas

DISCLAIMER

$$\hbar = 0$$

Purely classical (stochastic) world

CRASH COURSE ON LARGE DEVIATIONS

- **Large deviation theory** deals with the **exponential decay of probabilities** in stochastic processes

- **Example**: tossing N coins → sequence $(b_1, \ldots, b_N)$ with $b_i = 0$(tail), 1(head) and P(0)=P(1)=1/2. Fraction of heads:

$$R_N = \frac{1}{N} \sum_{i=1}^{N} b_i$$

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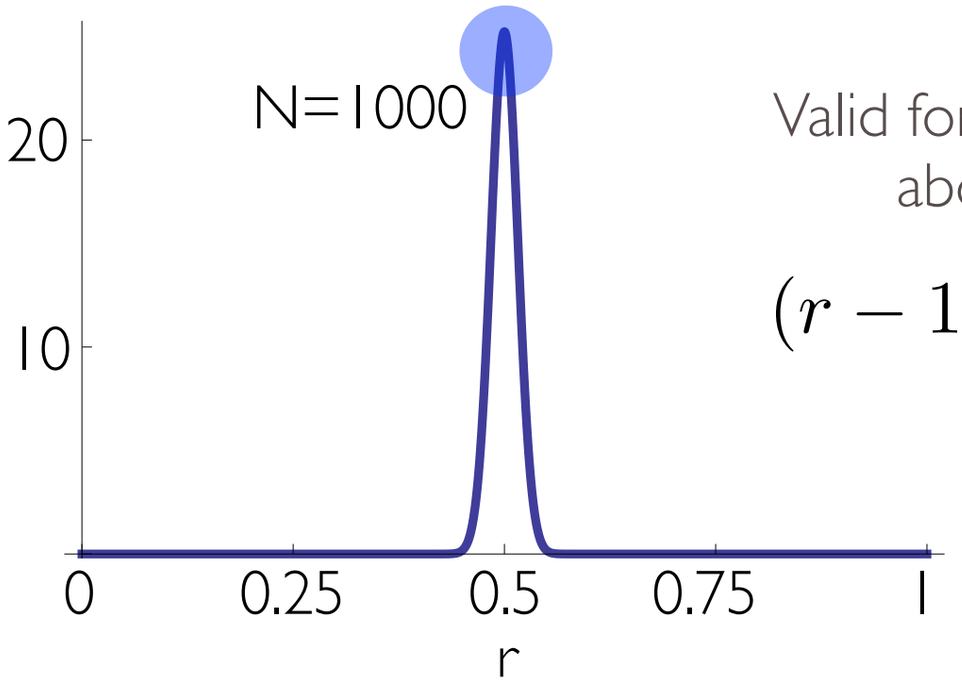
- Good approx: **Central Limit Theorem** $P(R_N = r) \approx e^{-\frac{(r-\mu)^2}{2\sigma^2/N}} = e^{-N2(r-\frac{1}{2})^2}$



Valid for small fluctuations about the mean

$$(r - 1/2) \sim \mathcal{O}(1/\sqrt{N})$$

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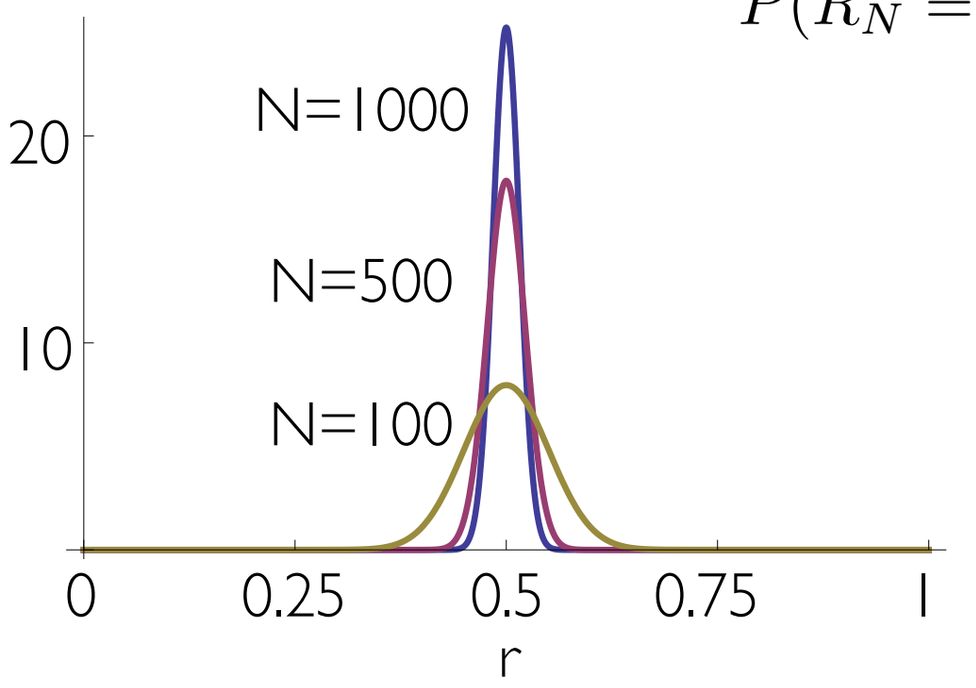

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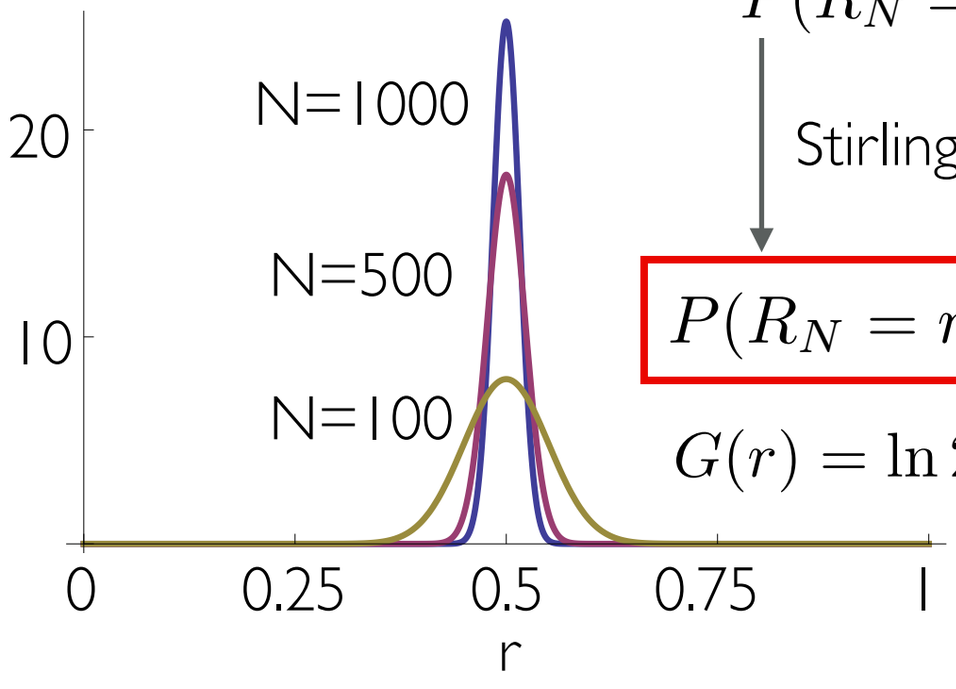


$$P(R_N = r) = \frac{1}{2^N}\frac{N!}{(Nr)!((1-r)N)!}$$

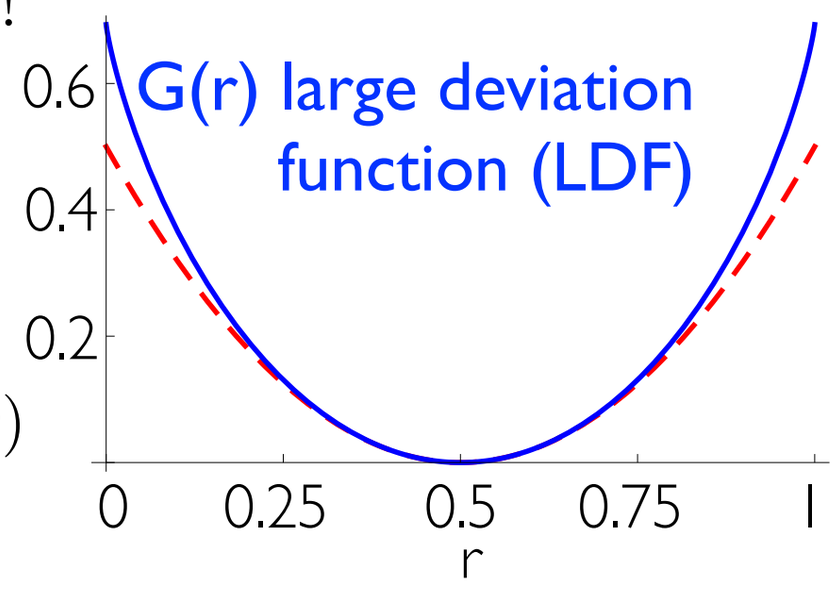
(binomial)

Stirling: $N! \approx N^N e^{-N}$

$$P(R_N = r) \approx e^{-NG(r)}$$

Large deviation principle

$$G(r) = \ln 2 + r\ln r + (1-r)\ln(1-r)$$



EQUILIBRIUM [Touchette Phys. Rep. 2009]

- **Large deviation theory** is the **mathematical language of Statistical Mechanics**

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- **Microcanonical ensemble:**

  - ☑ Number of microstates

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- **Canonical ensemble:**

  - Partition function:

$$Z(\beta) = \langle e^{-\beta N u} \rangle \approx e^{-N f(\beta)}$$

  - The equilibrium macrostate ($N \to \infty$) is the one minimizing the **free energy**

$$f(\beta) = \min_u \{\beta u - s(u)\}$$

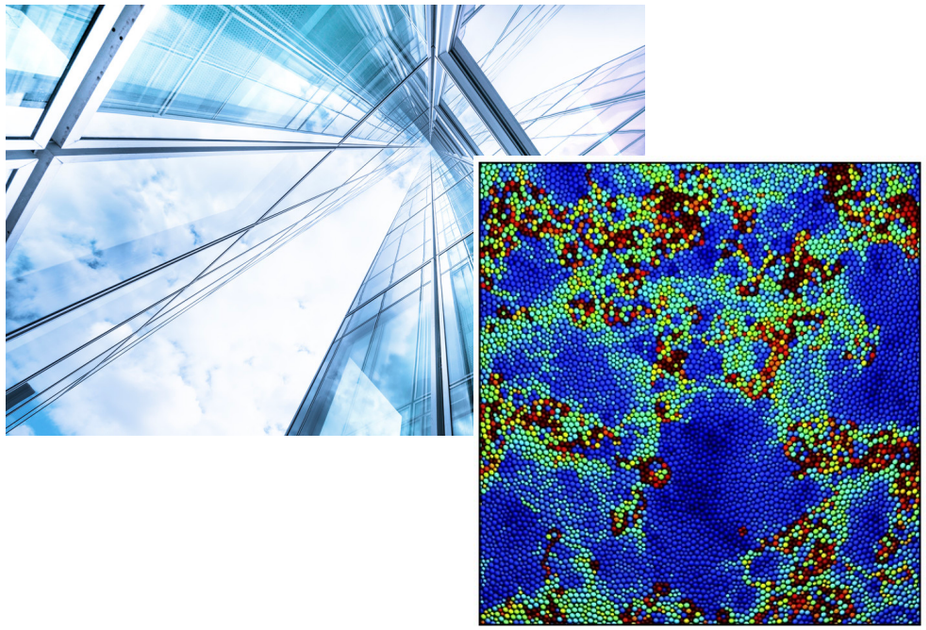
$$s(u) = \min_\beta \{\beta u - f(\beta)\}$$

OUT OF EQUILIBRIUM [Derrida 2007, Bertini el at 2001-2015]

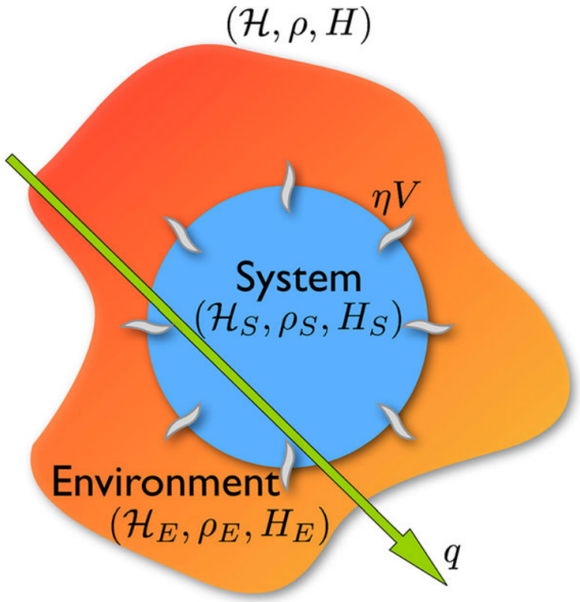
- Large deviation techniques can be applied to study properties of **nonequilibrium systems where equilibrium statistical mechanics cannot be applied**

Transport phenomena

Glassy systems



Open quantum systems



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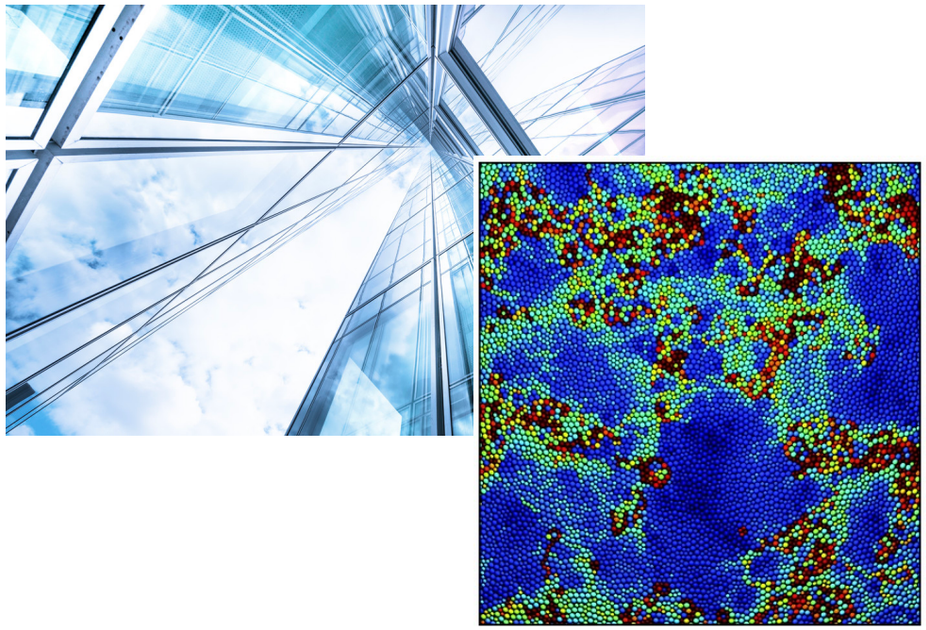

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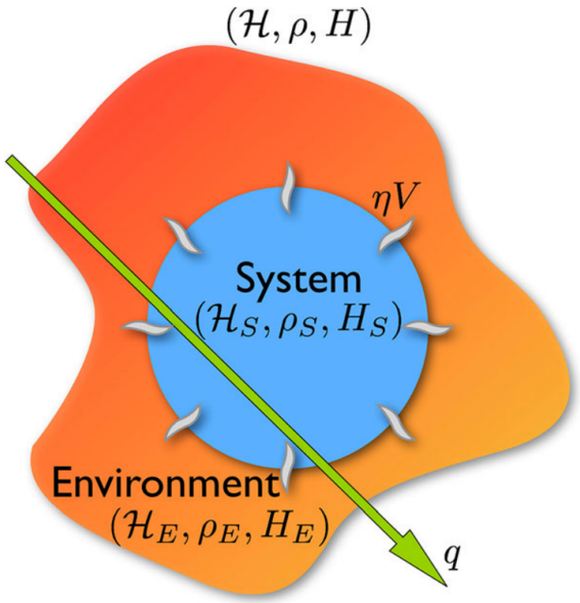
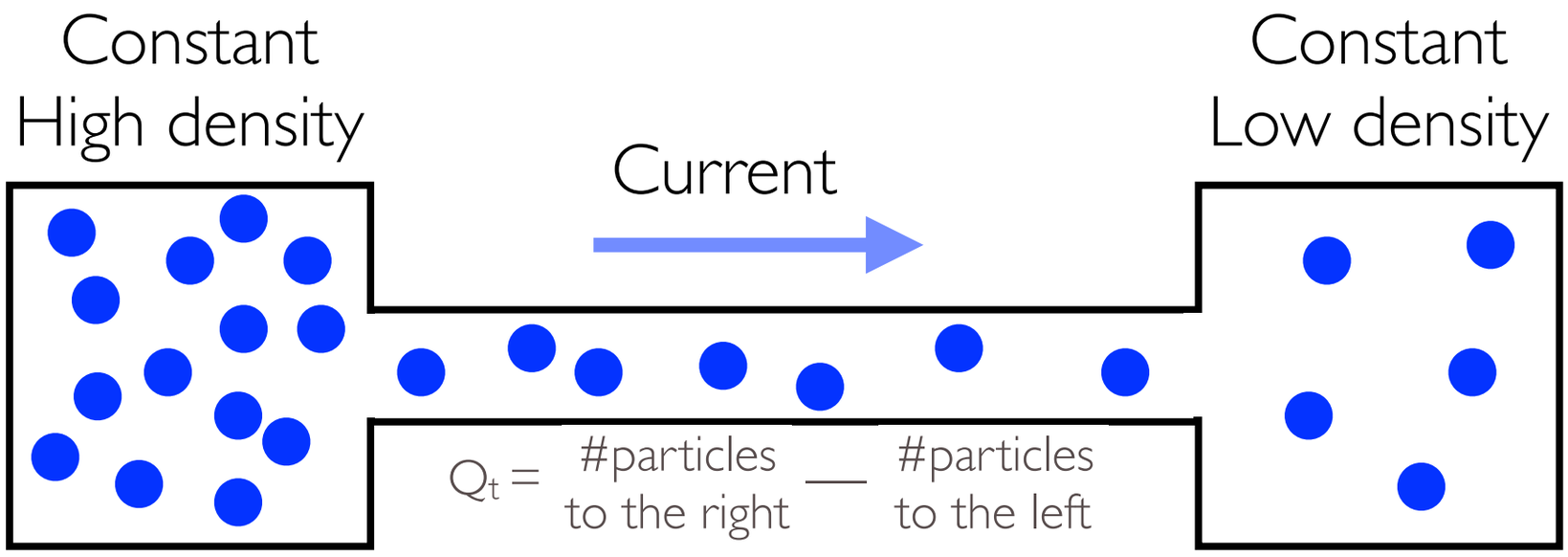


- Far from equilibrium **spatiotemporal correlations** emerge

- Interesting behavior is not in the configurations but in the **TRAJECTORIES**

NONEQUILIBRIUM FLUCTUATIONS



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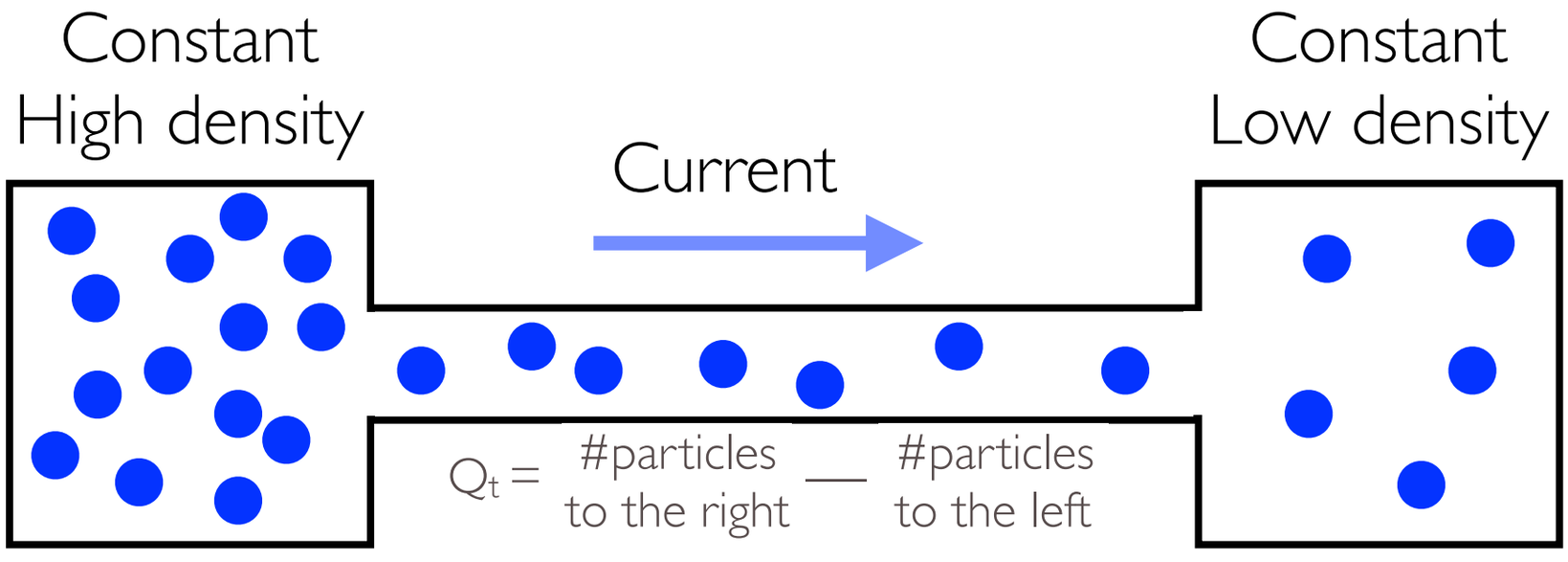

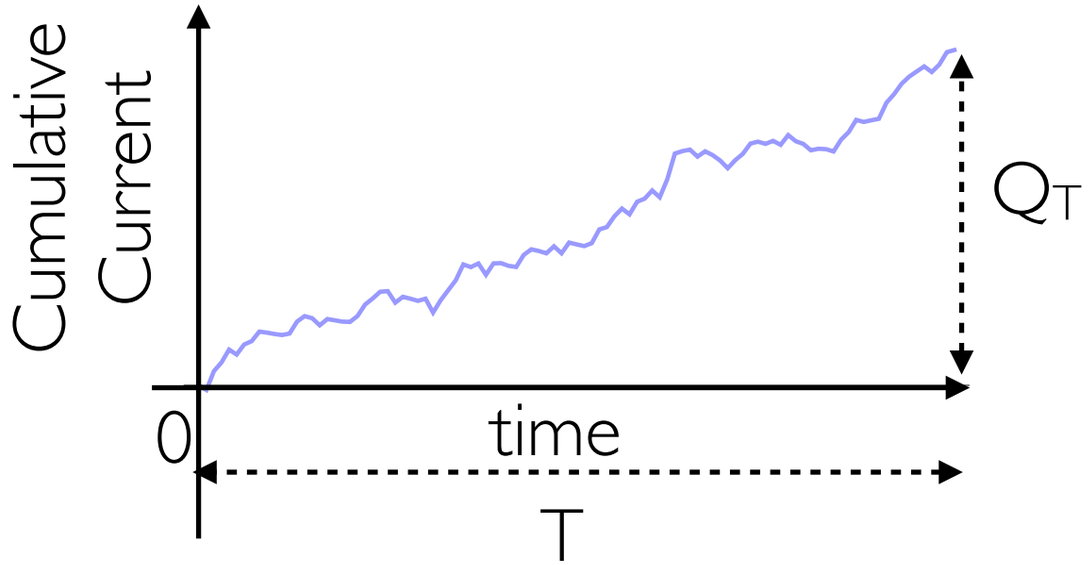

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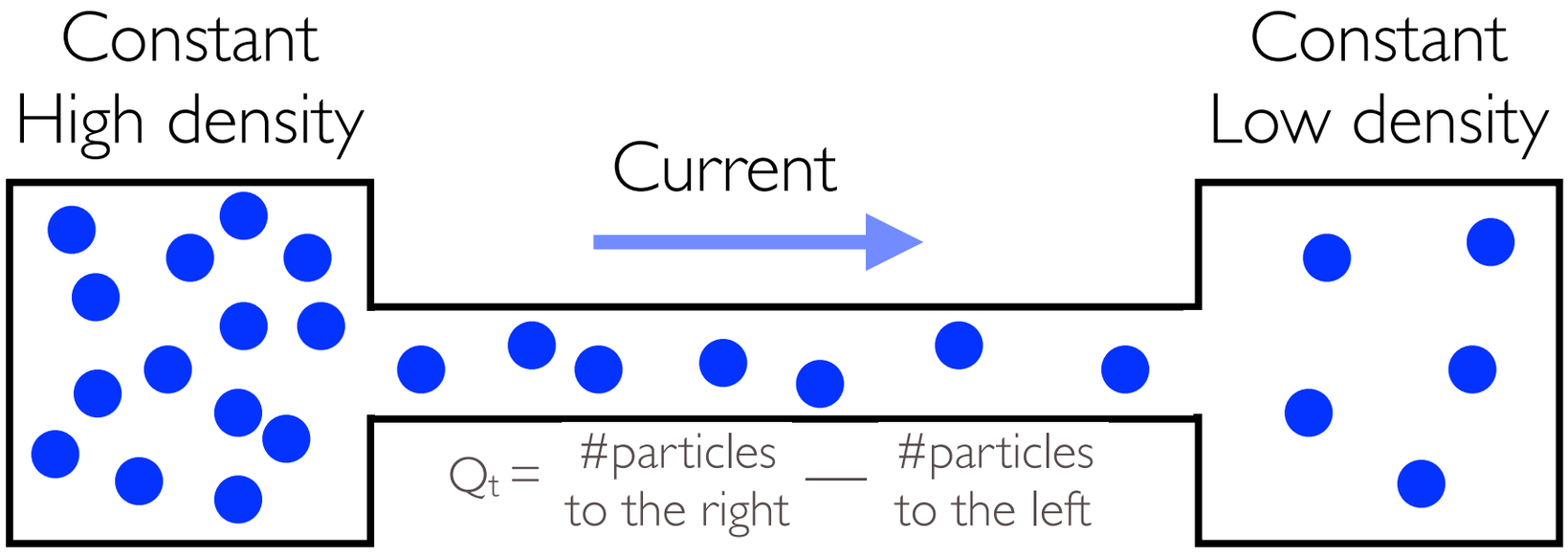
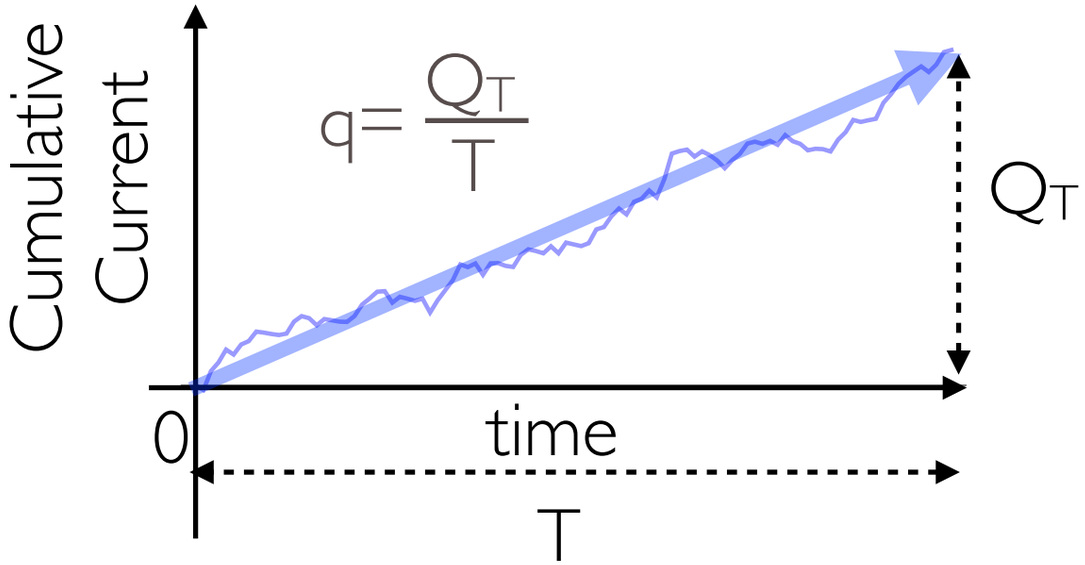


q : Dynamical observable



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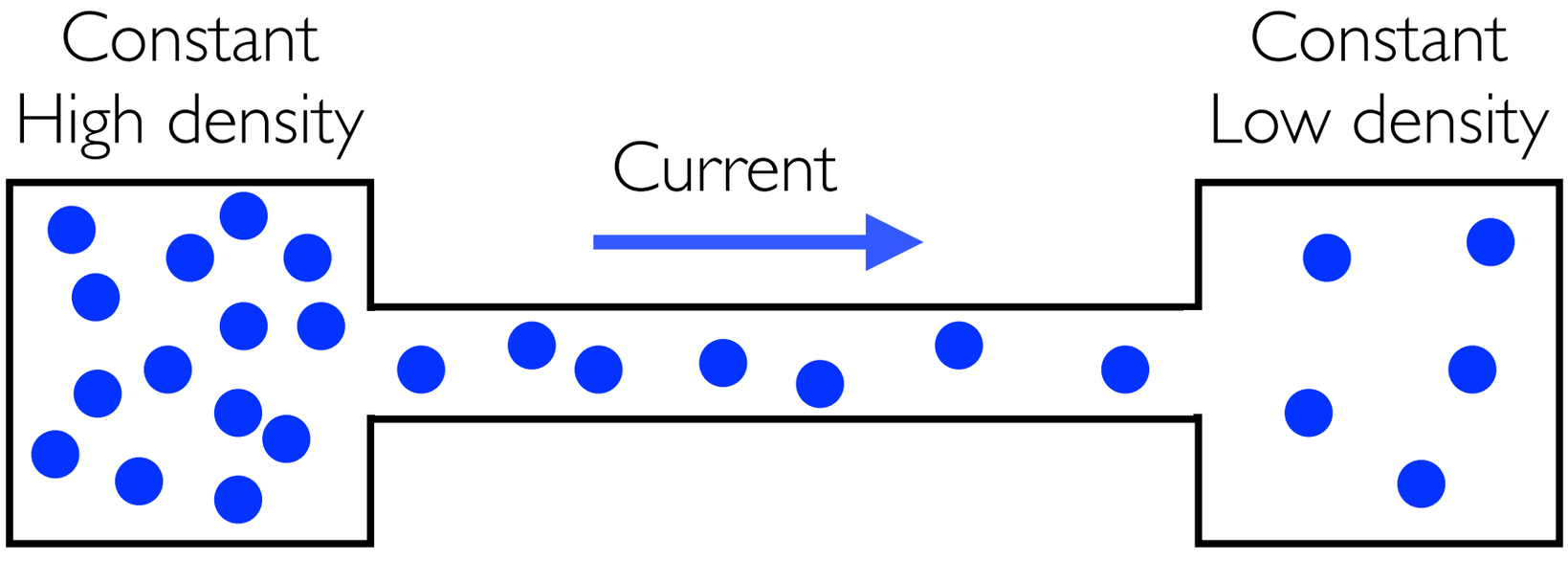

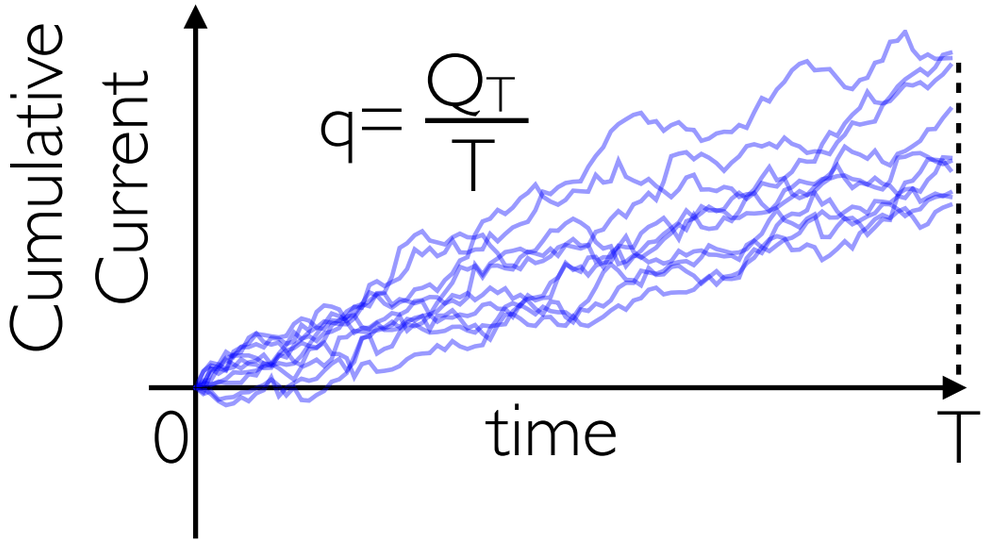

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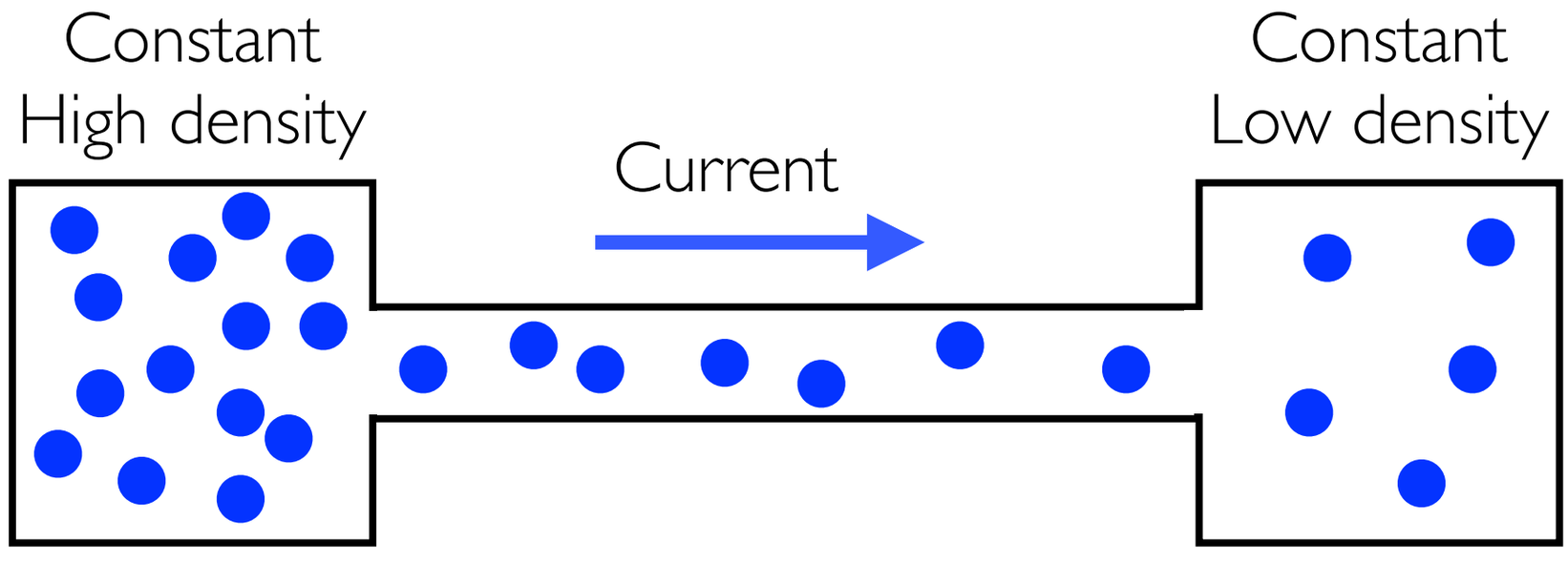

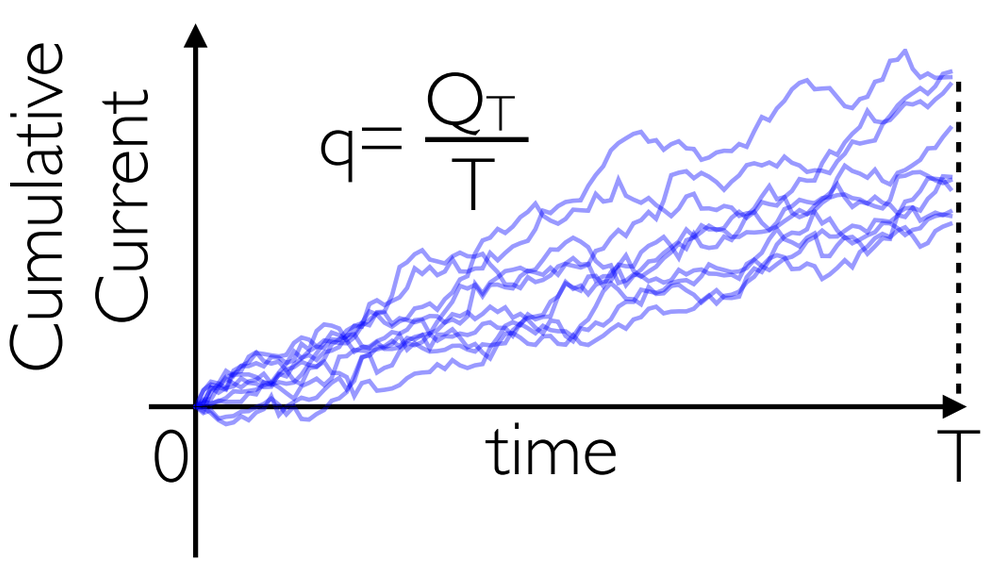

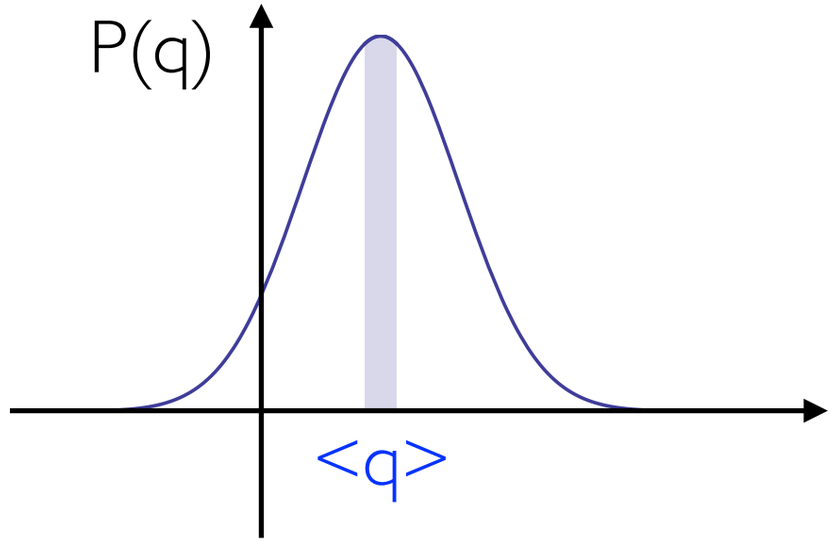

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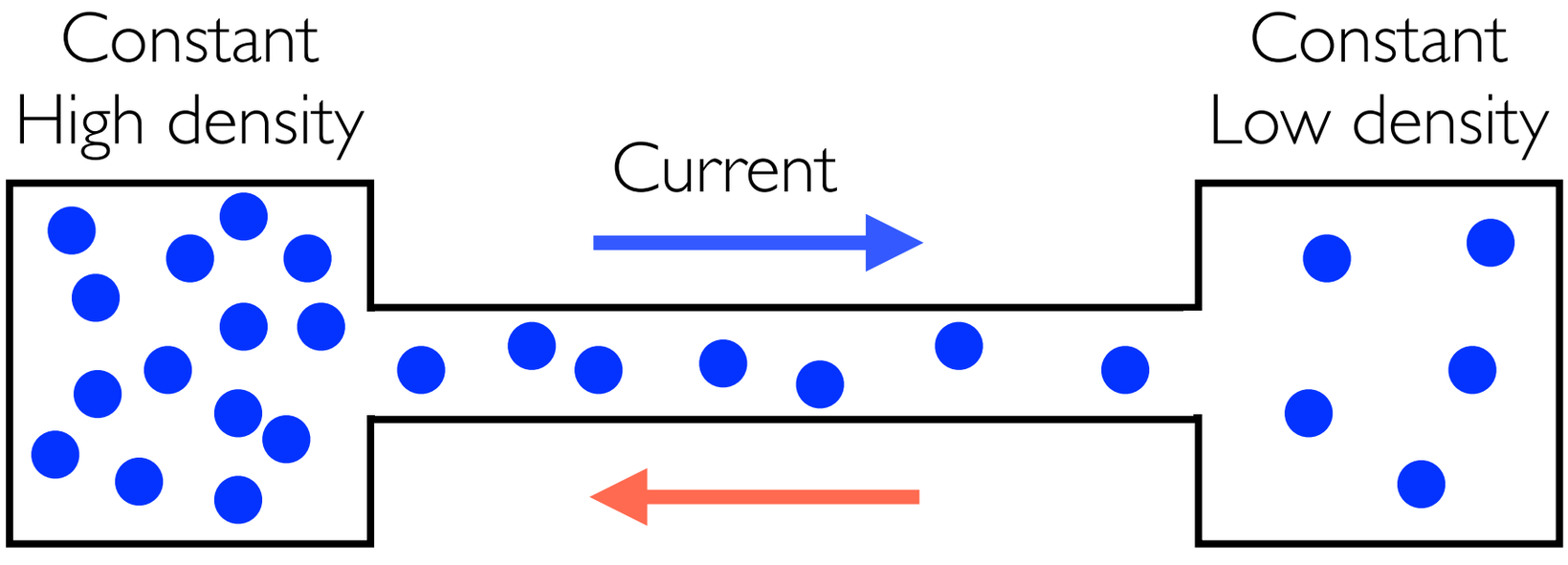

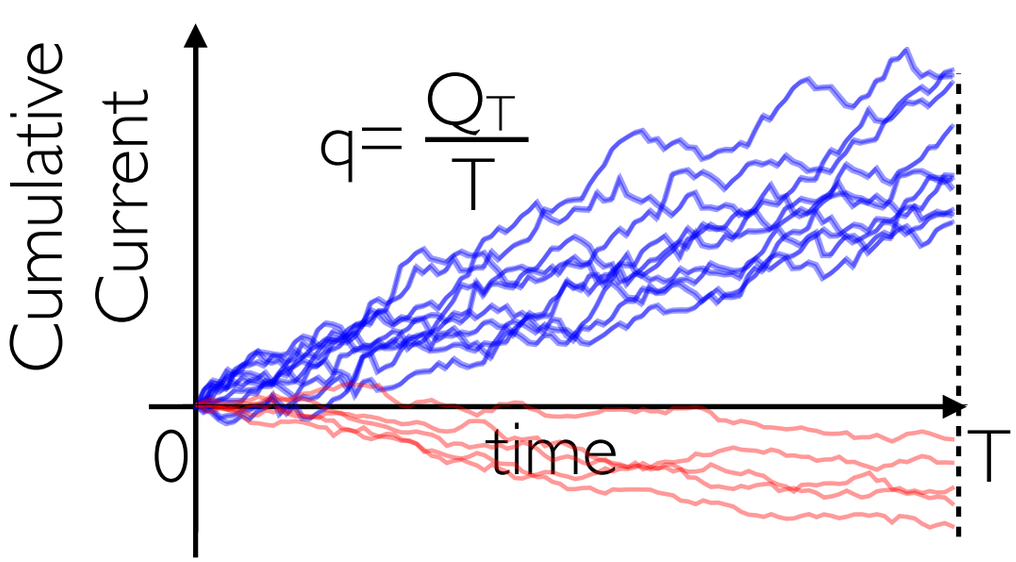

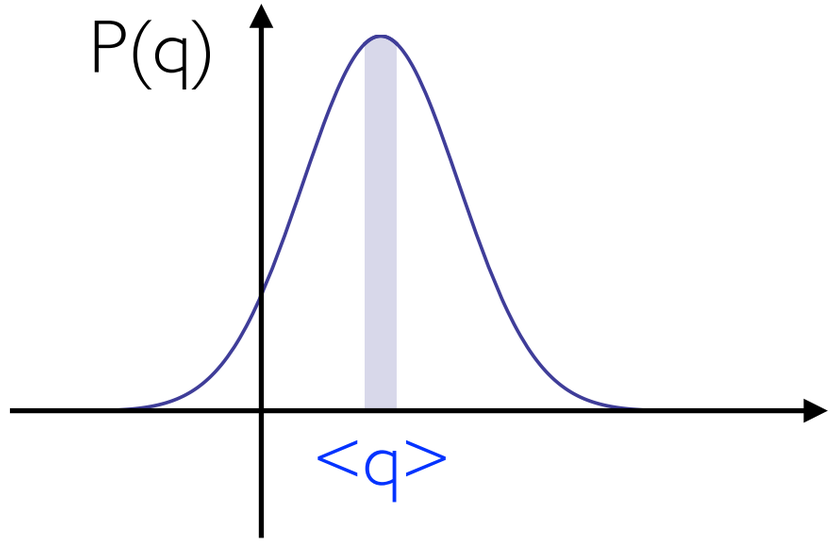

If we do MANY measurements…

negative currents!!

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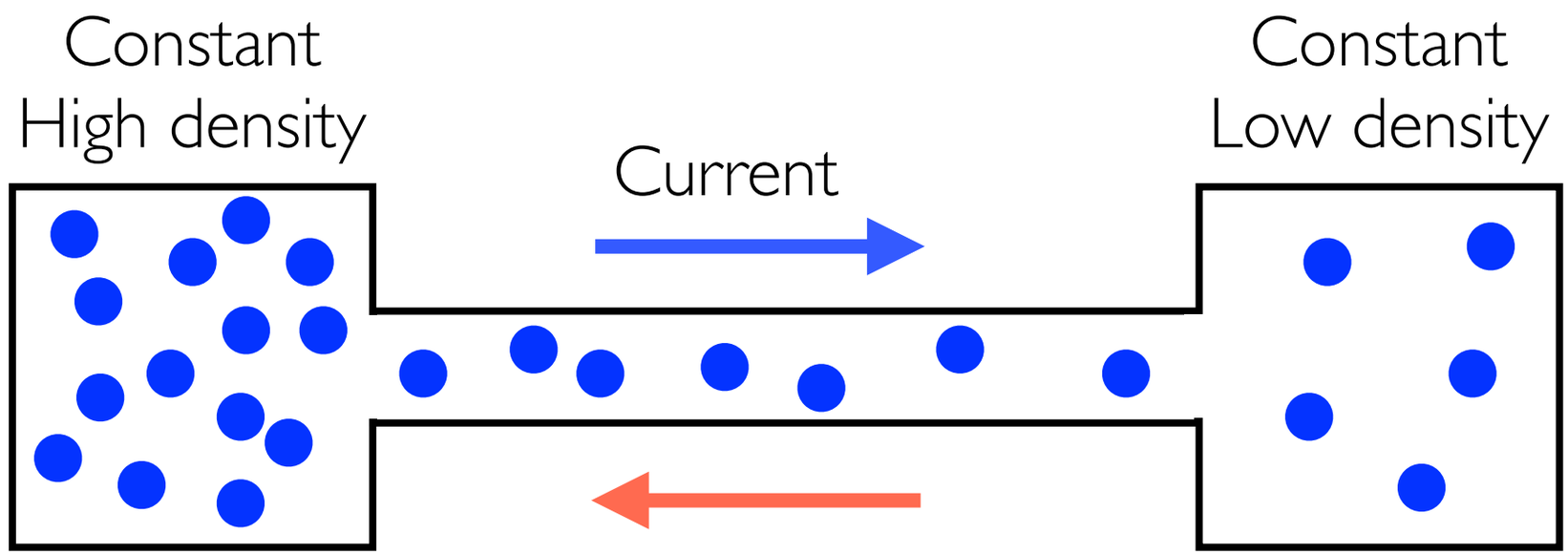
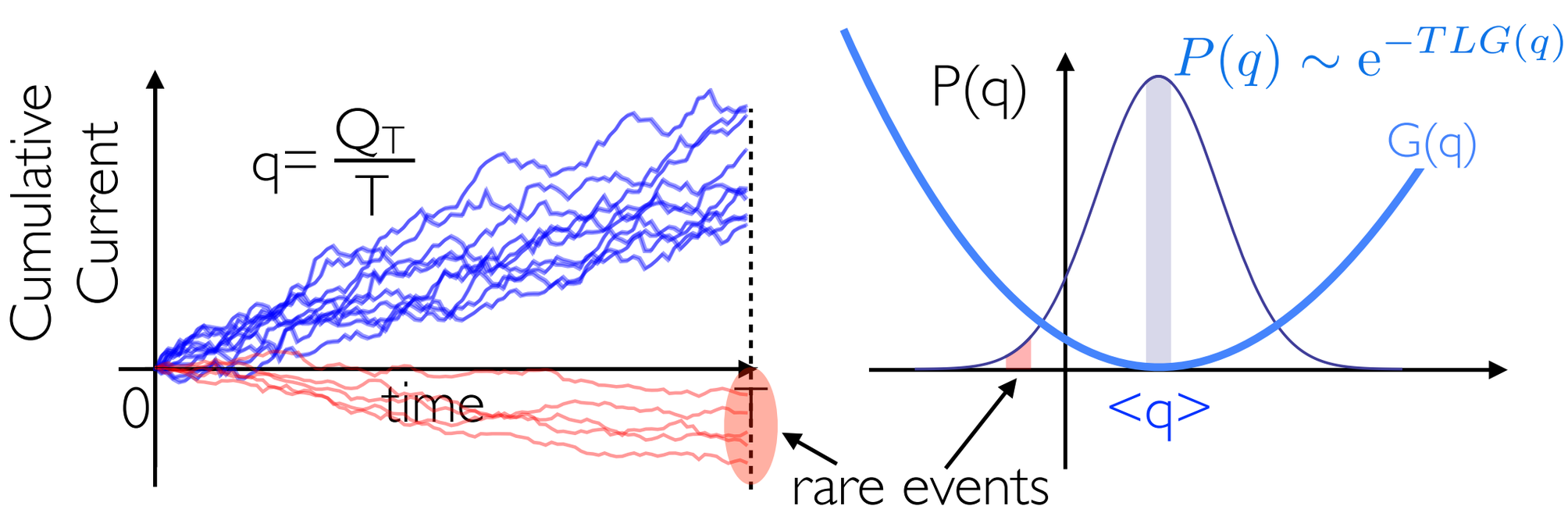




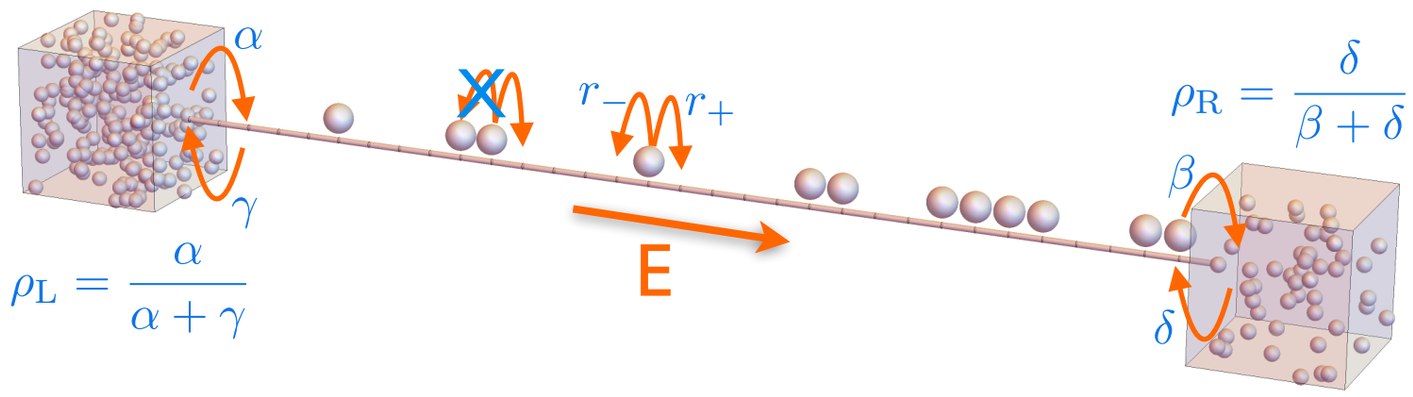
- The rate G(q) is a **large deviation function (LDF)**: it controls the statistics of fluctuations, and plays a **role akin to thermodynamic potentials out of equilibrium**

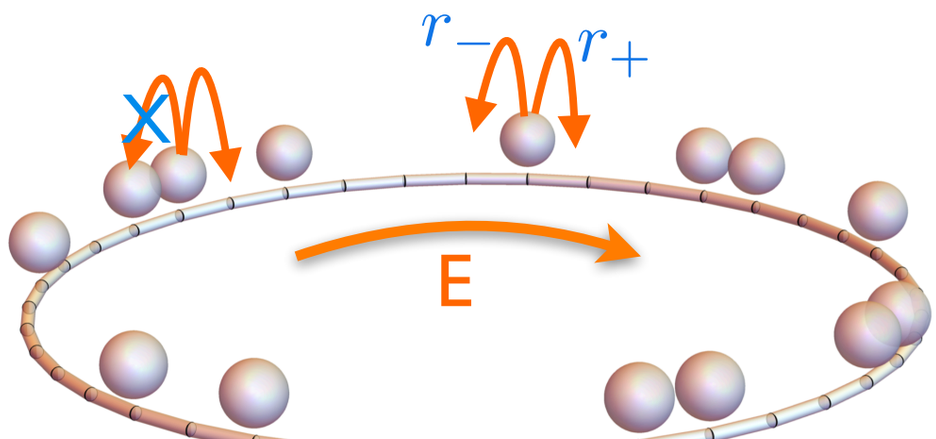
- Rare events have interesting physics, crucial to understand nonequilibrium physics

WEAKLY ASYMMETRIC SIMPLE EXCLUSION PROCESS

- **WASEP:** 1d lattice with occupation numbers n_i=0,1 + particle jumps to empty neighbors with rates

$$r_\pm \equiv \frac{1}{2}\exp(\pm E/L)$$





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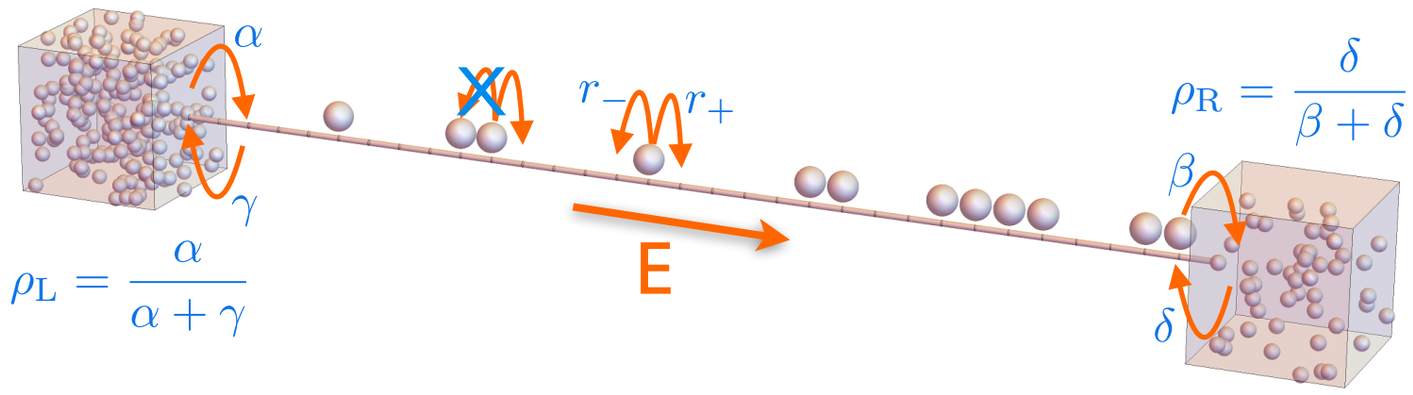

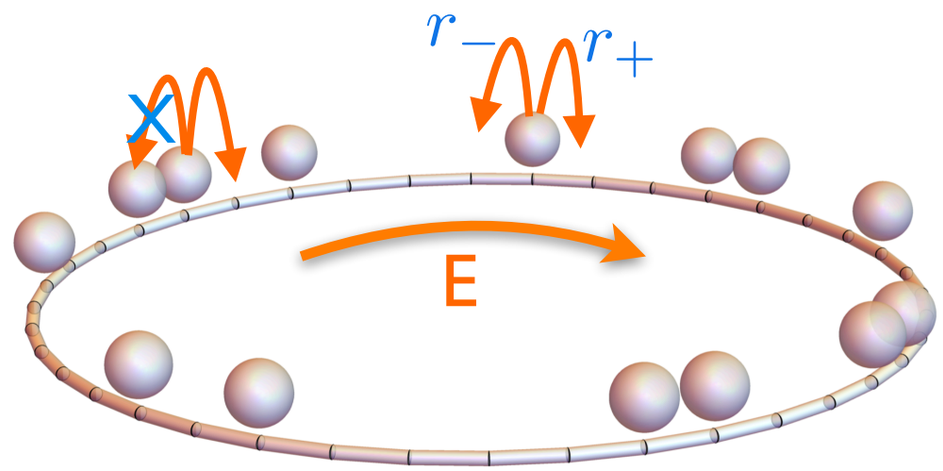

- At the **mesoscopic scale**, driven diffusive system

$$\partial_t \rho = -\partial_x \Big(-D(\rho)\partial_x \rho + \sigma(\rho)E + \sqrt{\sigma(\rho)}\xi(x,t) \Big)$$

$$D(\rho) = 1/2$$

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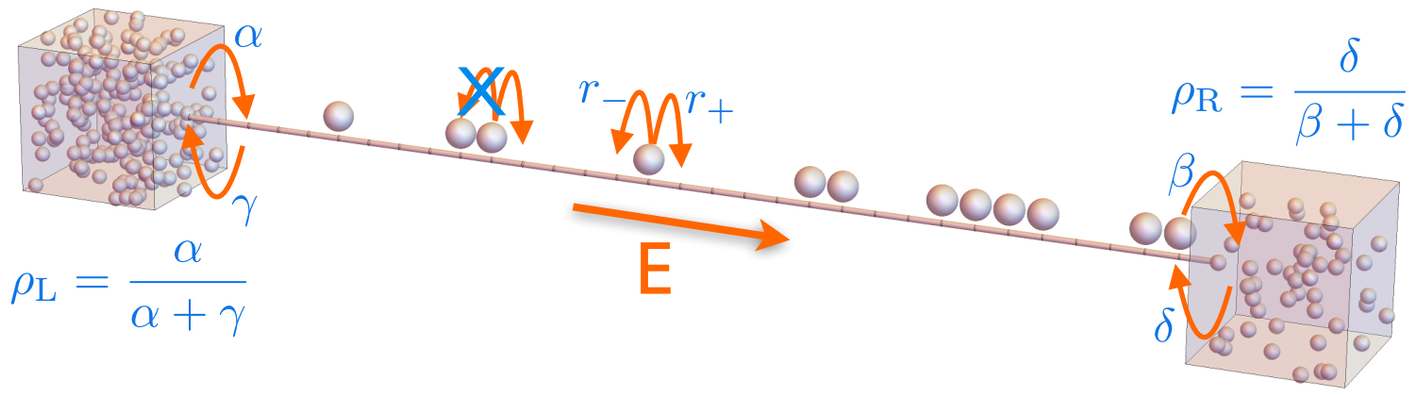

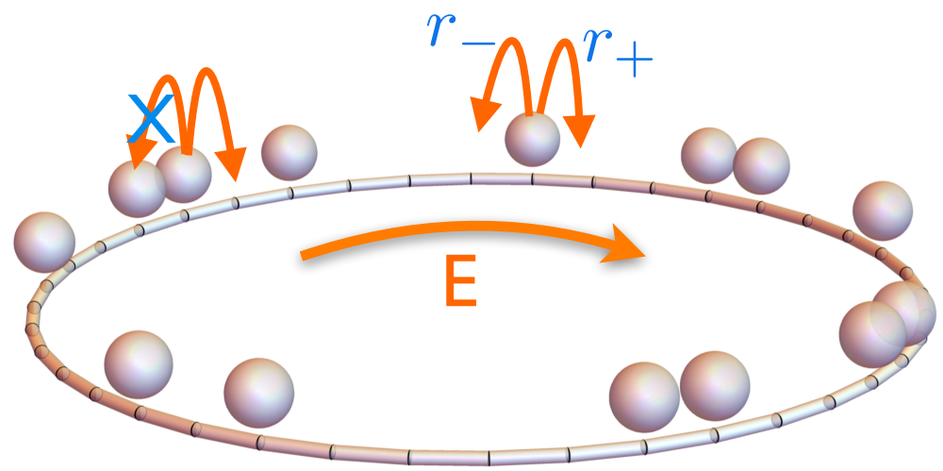

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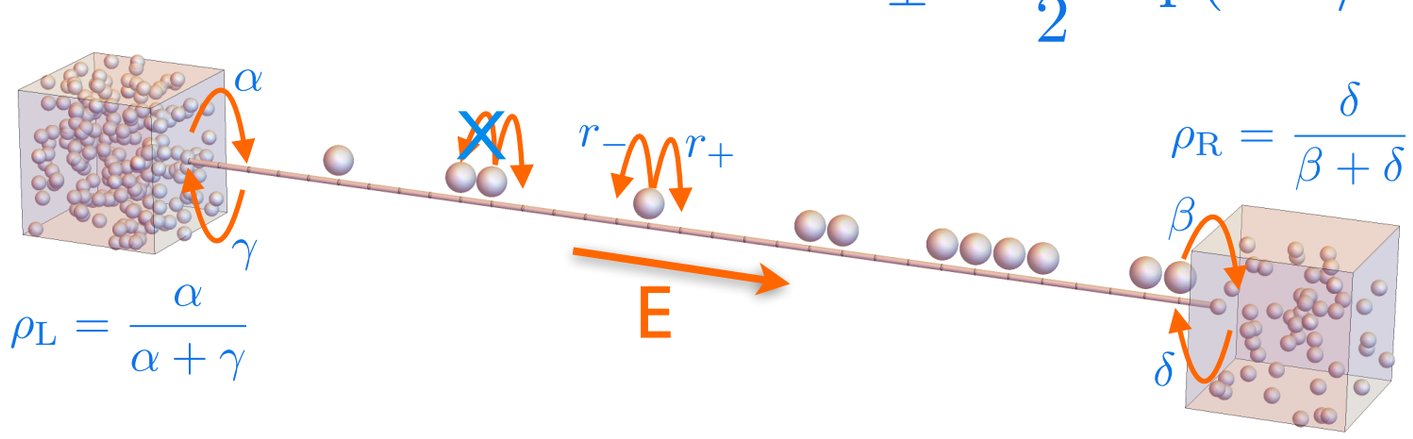

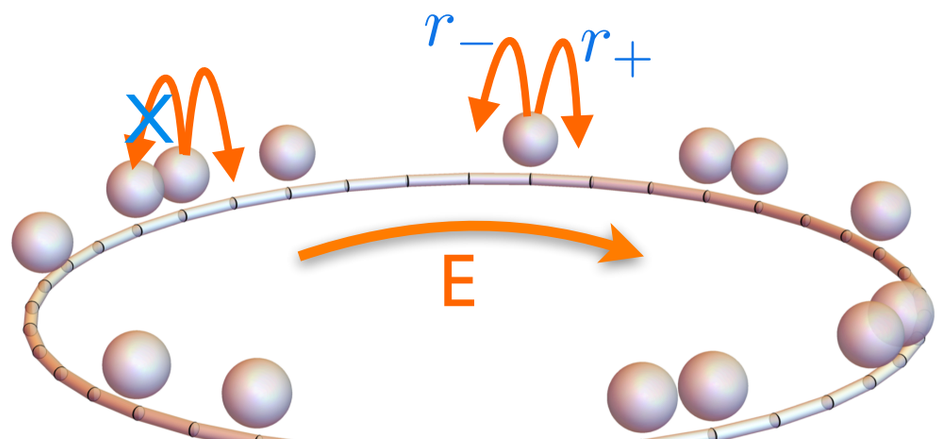

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- **Macroscopic fluctuation theory** (**MFT**) [Bertini, Jona-Lasinio et al Rev. Mod. Phys. 2015]

$$P(\{\rho, j\}_0^\tau) \sim \mathrm{e}^{-L\,\mathcal{I}_\tau[\rho,j]}$$

$$\mathcal{I}_\tau[\rho,j] = \int_0^\tau dt \int_0^1 dx \frac{\Big(j + D(\rho)\partial_x\rho - \sigma(\rho)E\Big)^2}{2\sigma(\rho)}$$

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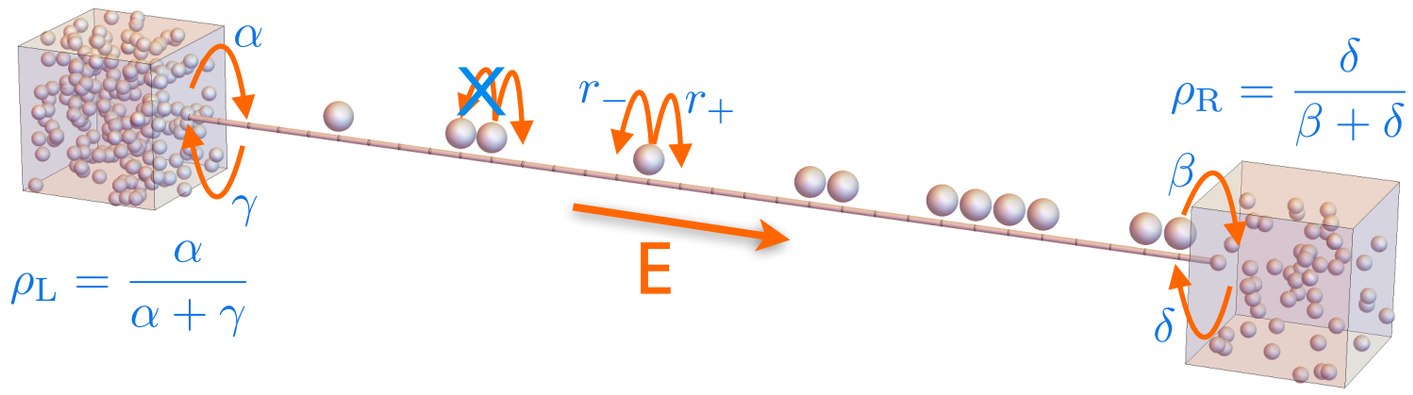

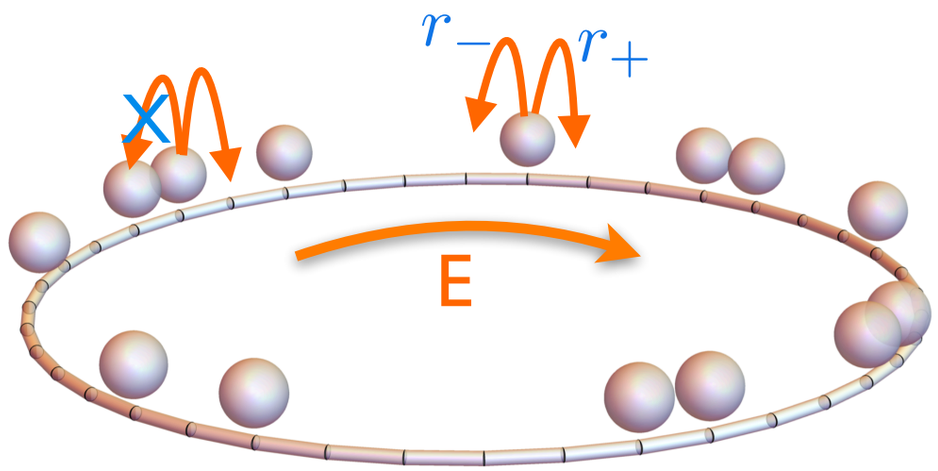

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$$P(\{\rho, j\}_0^\tau) \sim \mathrm{e}^{-L\,\mathcal{I}_\tau[\rho,j]} \quad \Big| \quad P(q) \sim \mathrm{e}^{-TLG(q)} \quad \Big| \quad G(q) = \lim_{\tau\to\infty} \tau^{-1} \min_{\{\rho,j\}_0^\tau} \mathcal{I}_\tau[\rho,j]$$

$$\mathcal{I}_\tau[\rho,j] = \int_0^\tau dt \int_0^1 dx \frac{\Big(j + D(\rho)\partial_x\rho - \sigma(\rho)E\Big)^2}{2\sigma(\rho)} \quad \Big| \quad q = \tau^{-1}\int_0^\tau dt \int_0^1 dx\, j(x,t)$$

DYNAMICAL PHASE TRANSITIONS

- **Dynamical phase transitions (DPTs)** may emerge **when conditioning a diffusive system to sustain an atypical current (or activity)**

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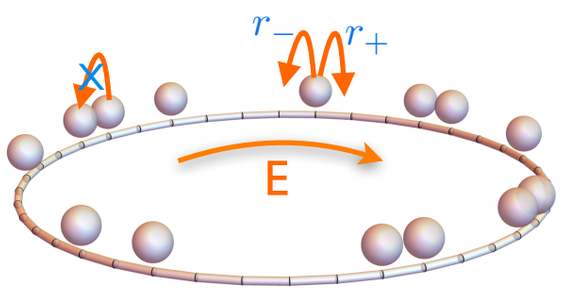
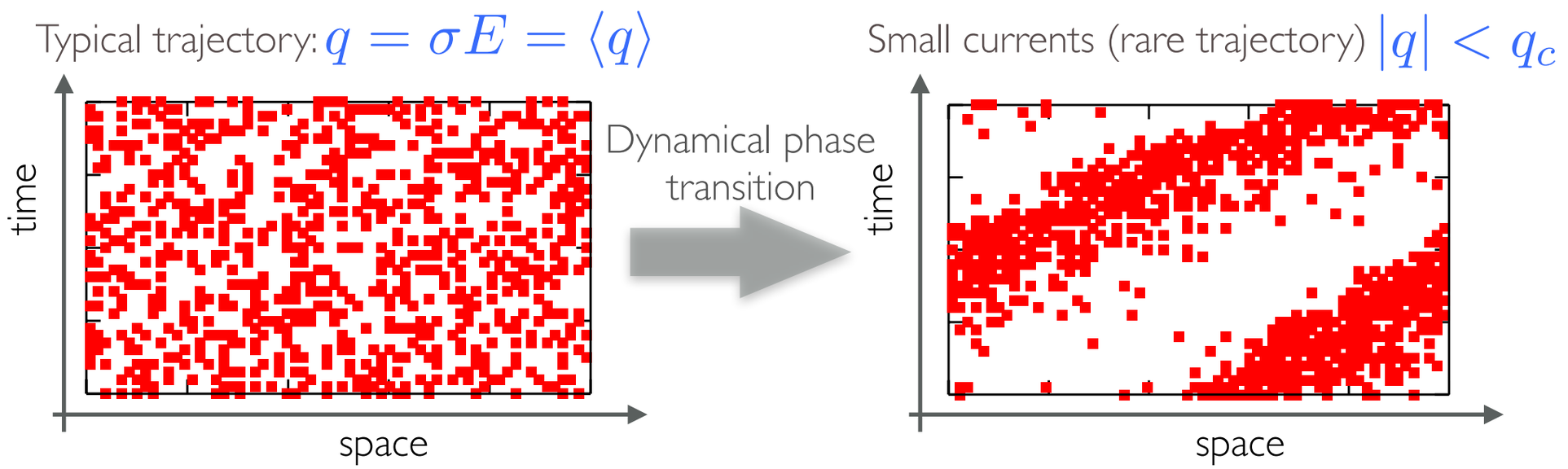
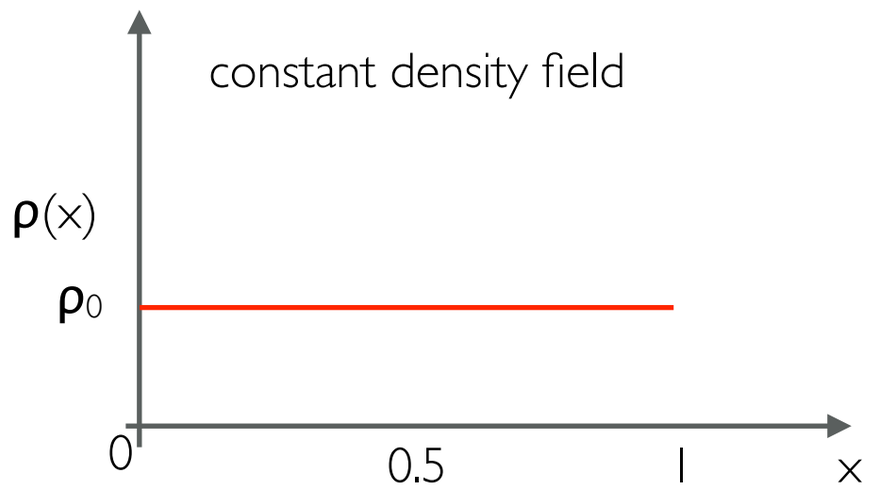
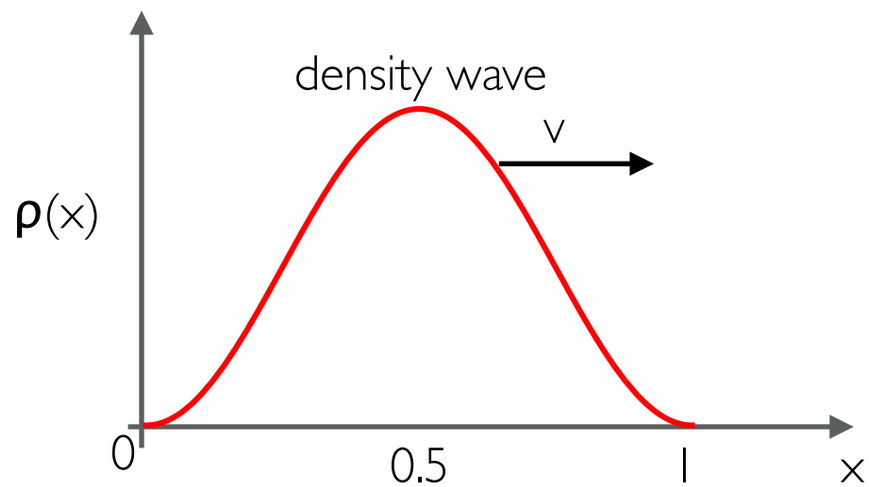
[Bodineau, Derrida PRE 2015]
[Espigares, Garrido, PH PRE 2013]

For WASEP









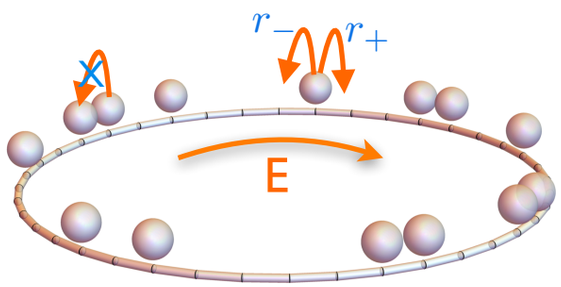
DYNAMICAL PHASE TRANSITIONS

- **Dynamical phase transitions (DPTs)** may emerge **when conditioning a diffusive system to sustain an atypical current (or activity)**

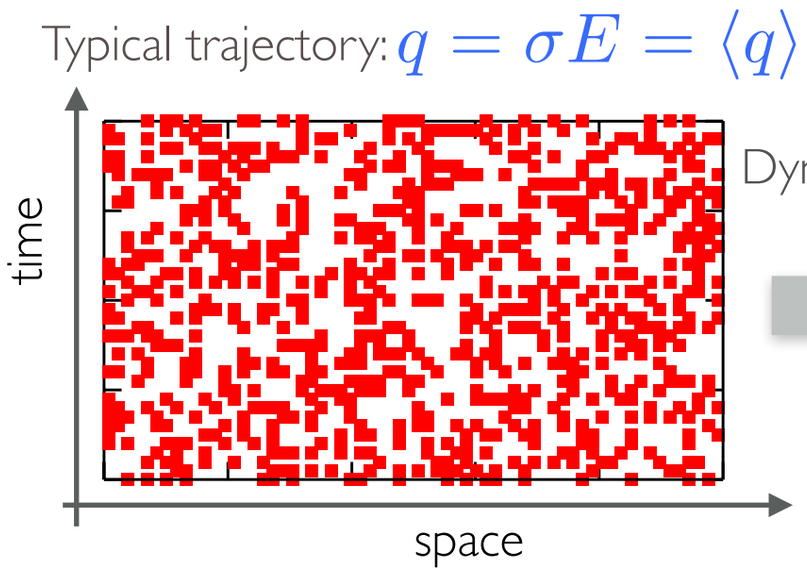
- **Dynamical phases** correspond to **different types of trajectories**: some may display **emergent order, collective rearrangements, and spontaneous symmetry-breaking**

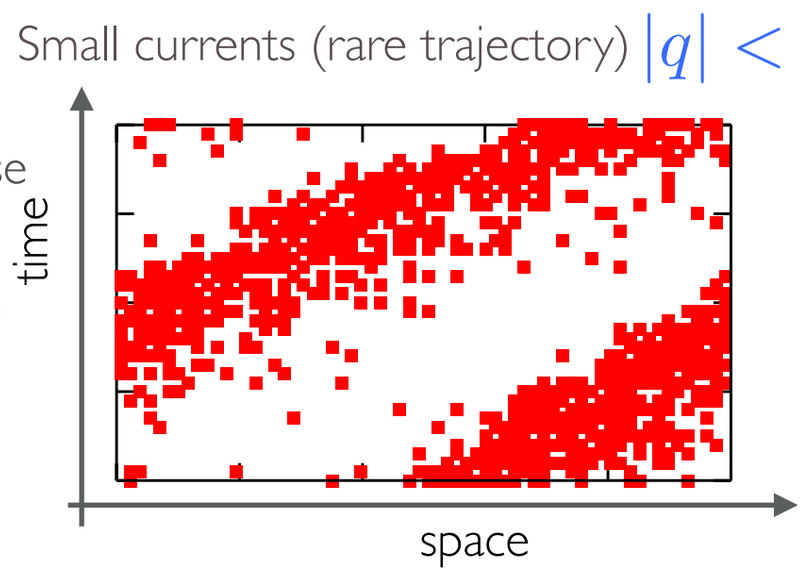
[Bodineau, Derrida PRE 2015]
[Espigares, Garrido, PH PRE 2013]

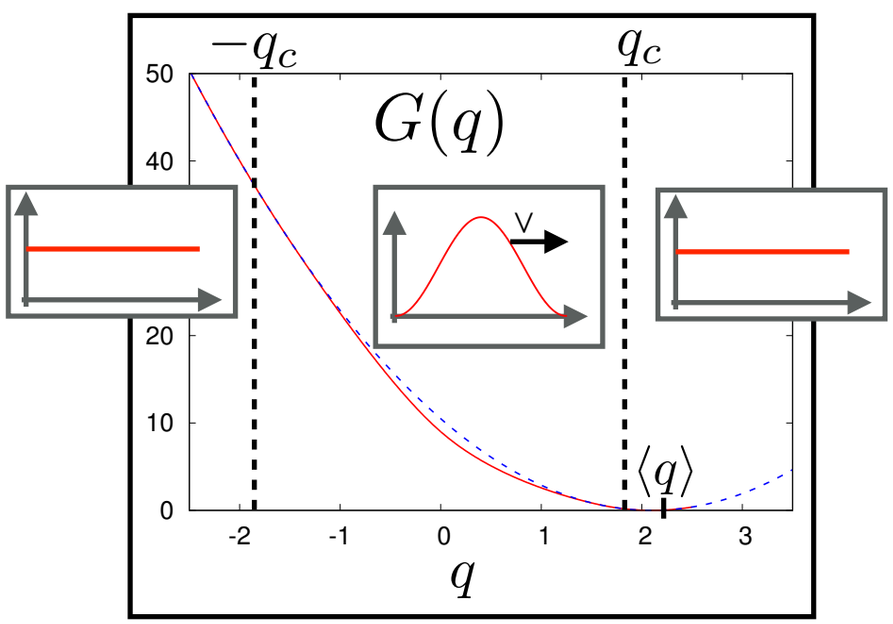
For WASEP

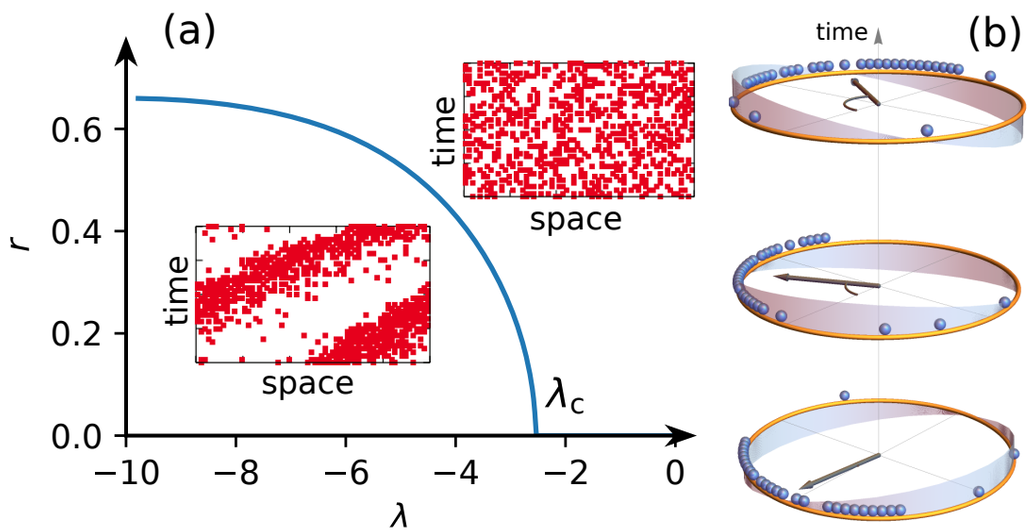


2nd-order DPT







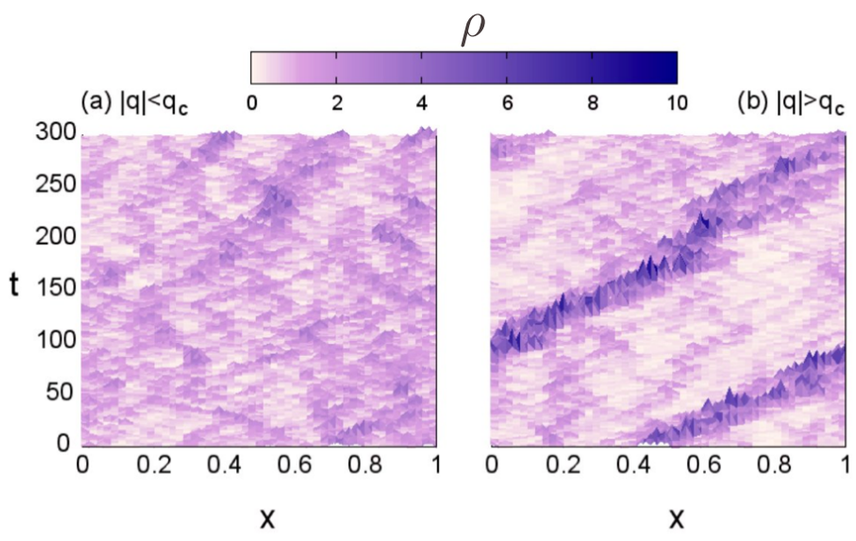


DYNAMICAL PHASE TRANSITIONS

- **Dynamical phase transitions abound** in the fluctuation of many-body systems

- A partial (and 100% biased) list of **examples**:

Energy diffusion in 1d



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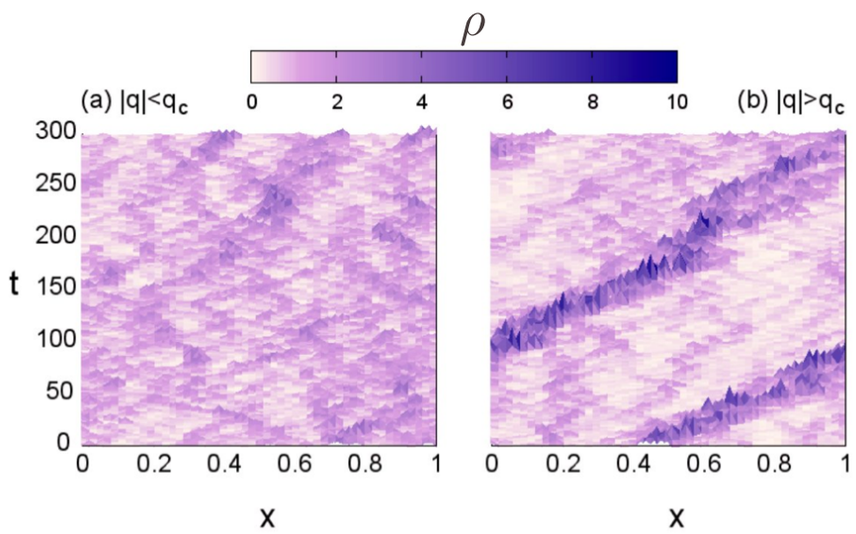

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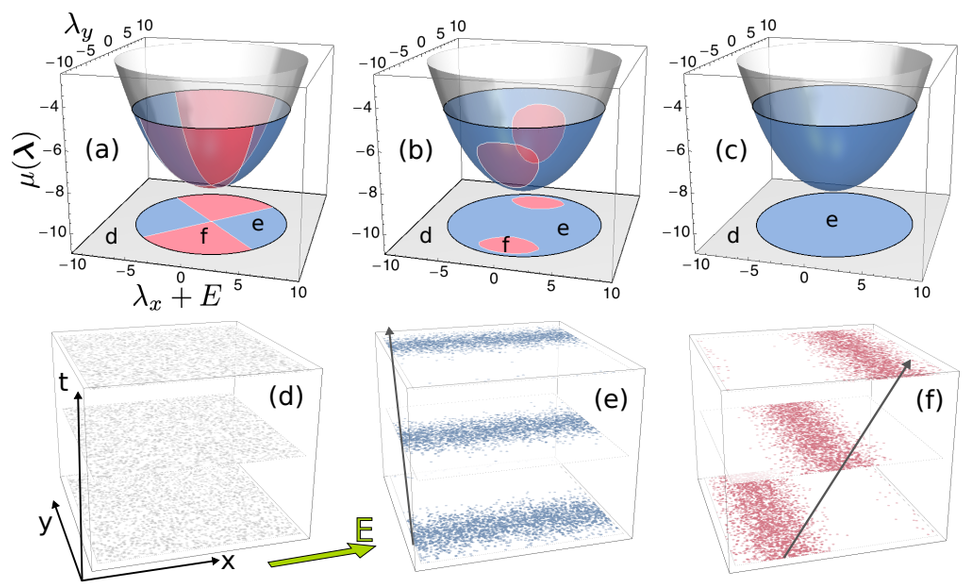
2d WASEP



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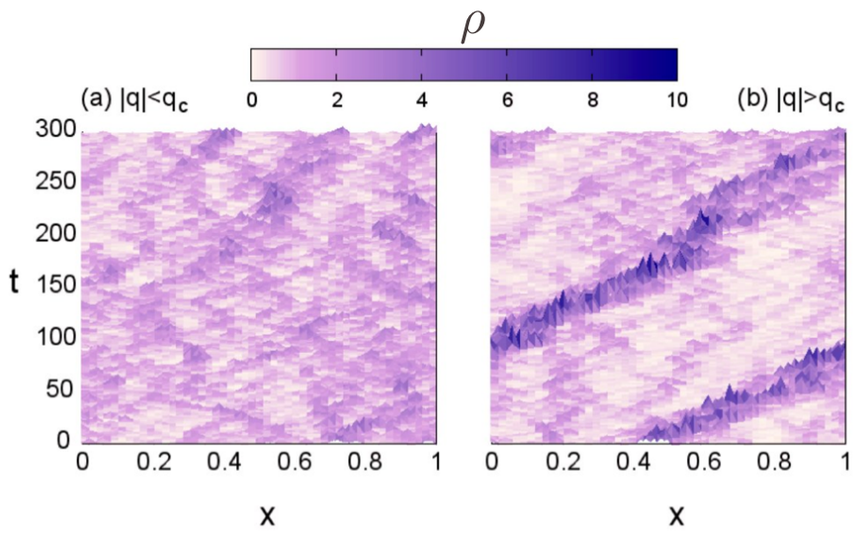

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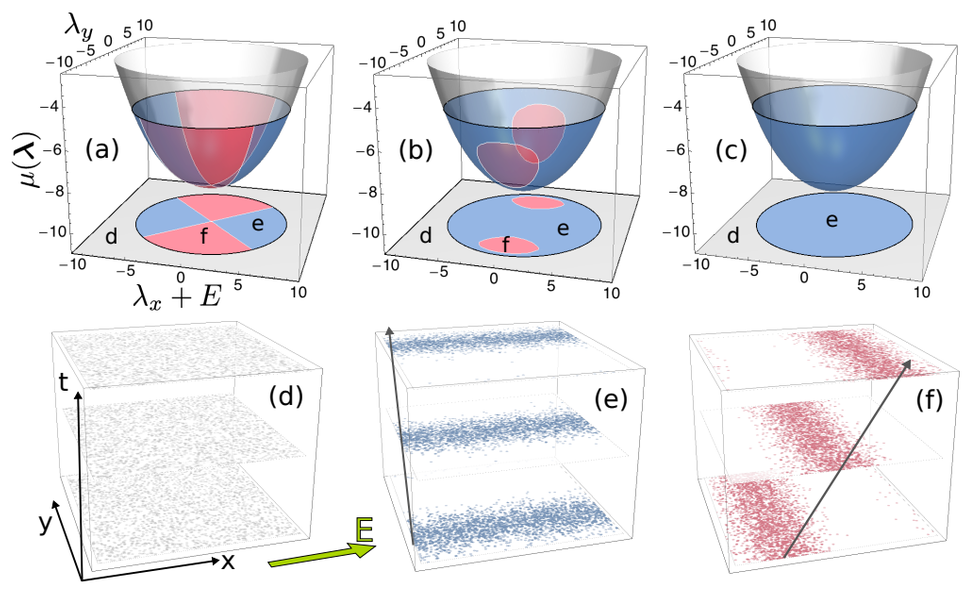

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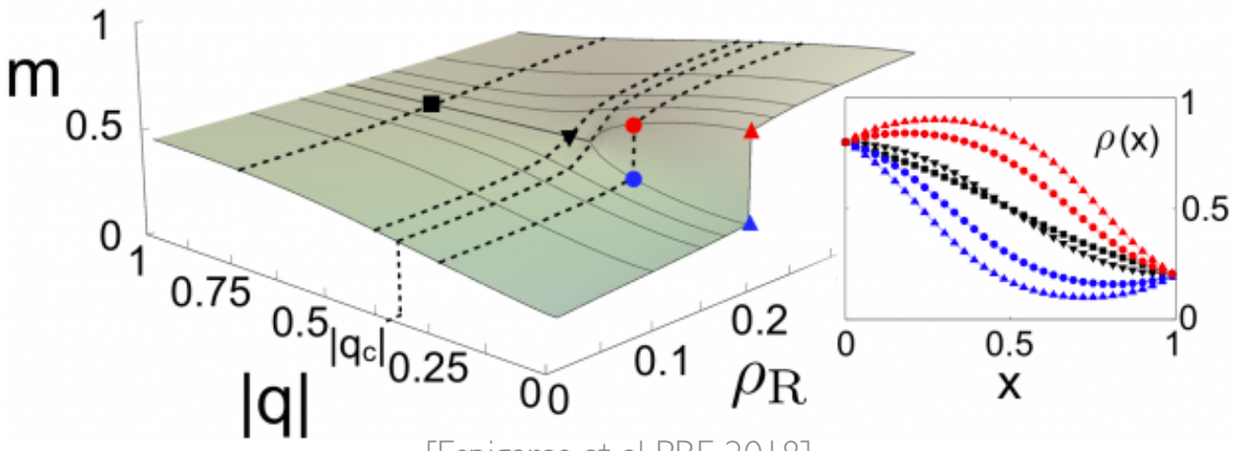
Open WASEP in 1d



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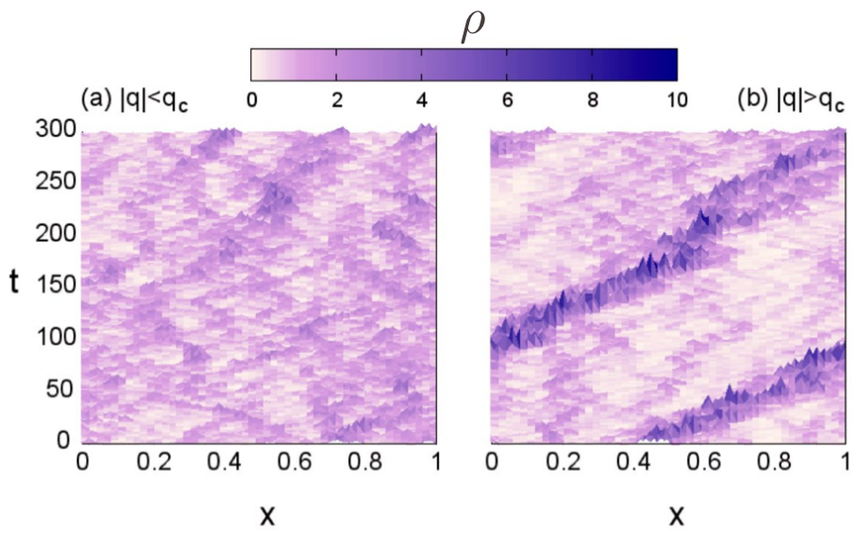

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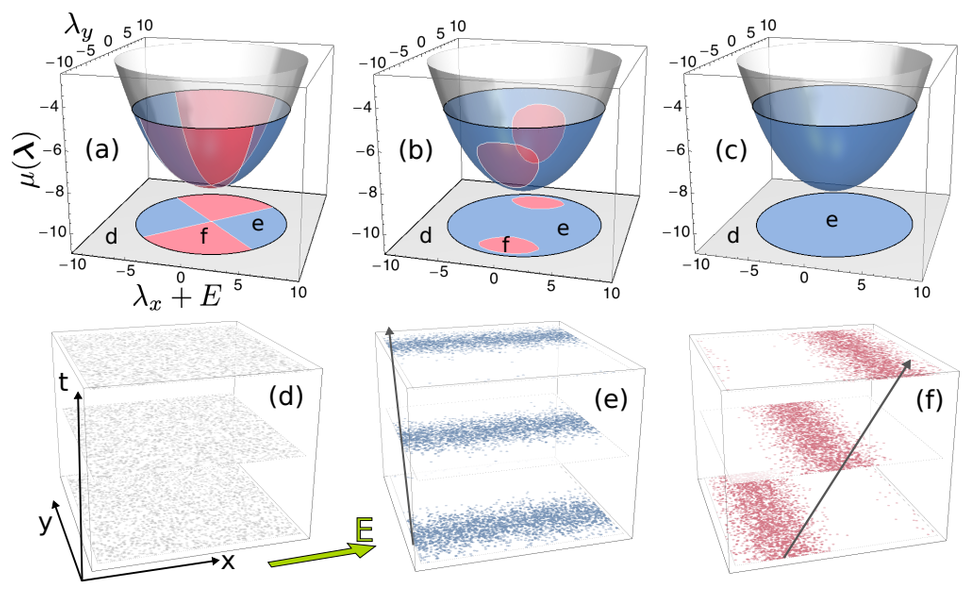

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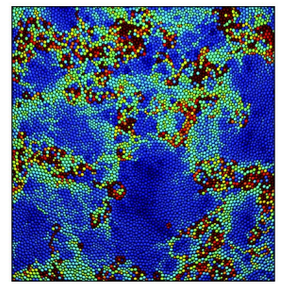

Activity DPT in glasses

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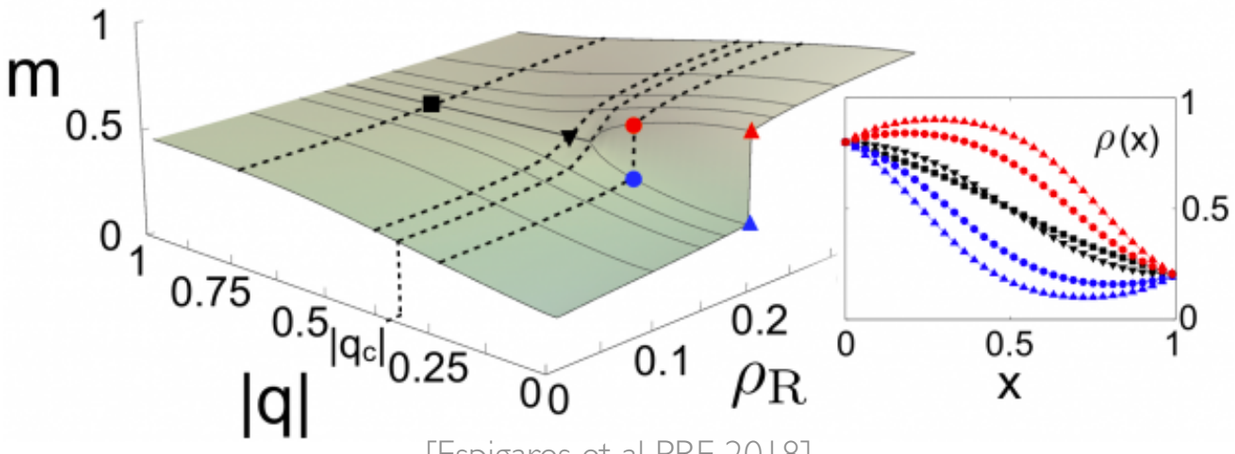

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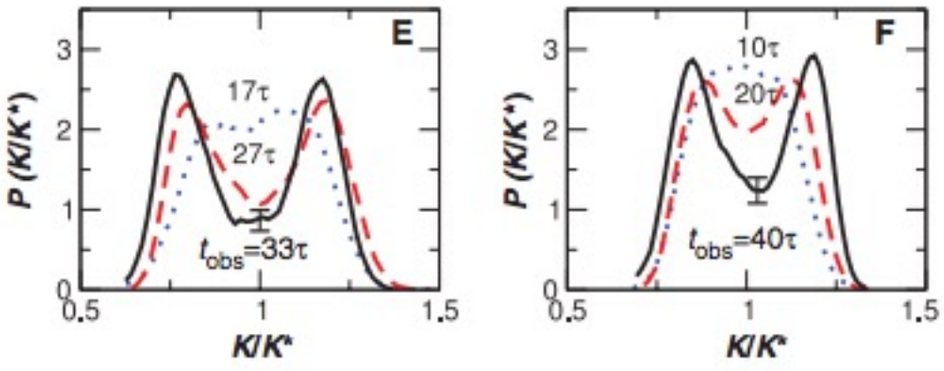

[Hedges et al Science 2009]
[Pinchaipat et al PRL 2017]

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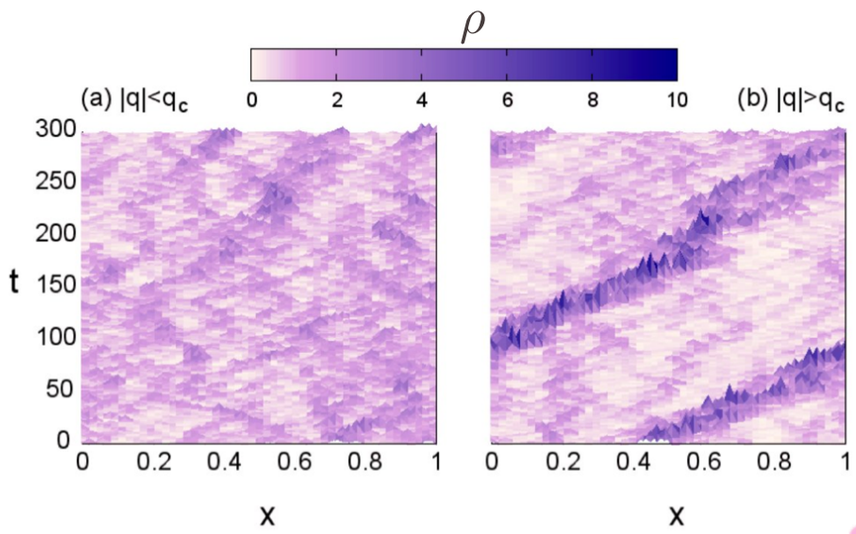

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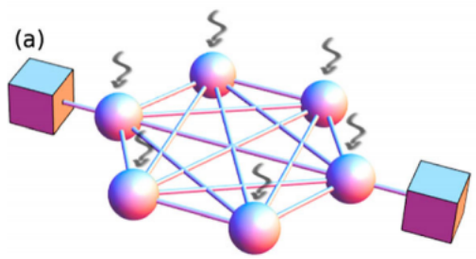
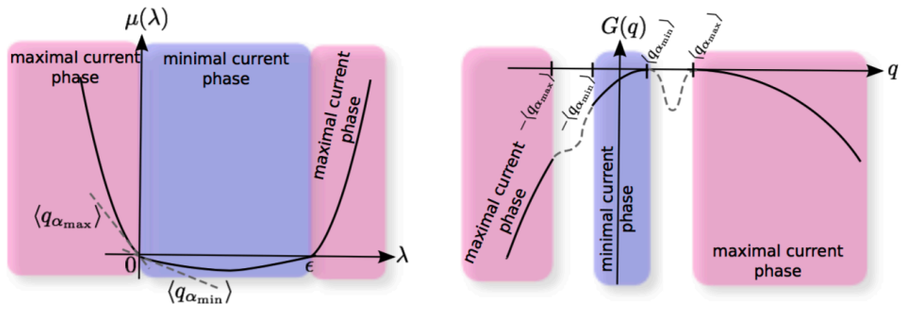
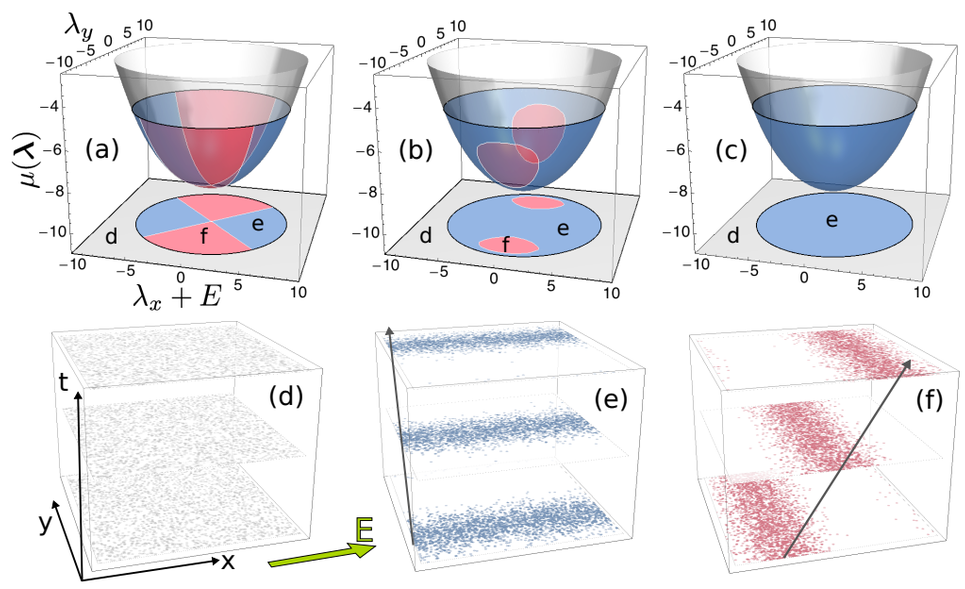


Symmetry-induced DPTs in open quantum systems



2d WASEP



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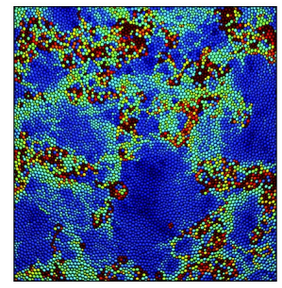
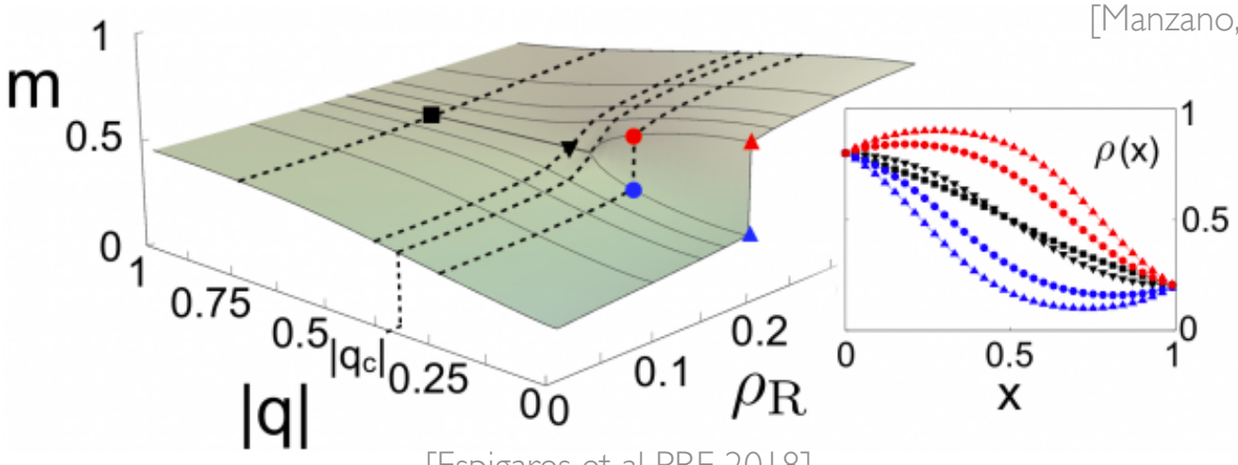


[Manzano, PH PRB 2014]

Open WASEP in 1d



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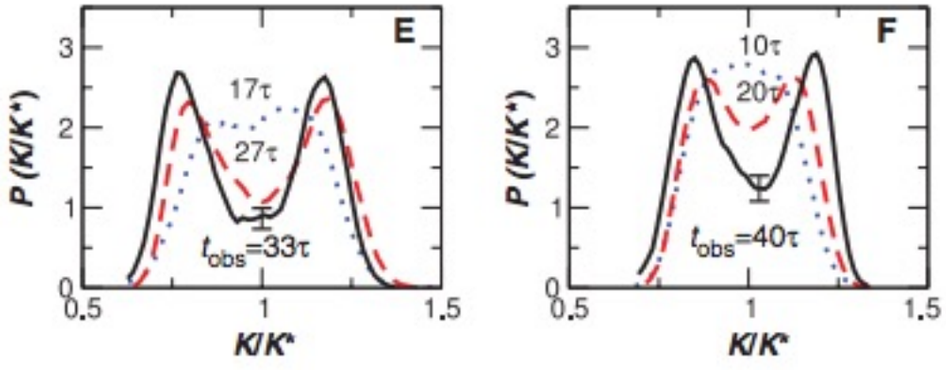

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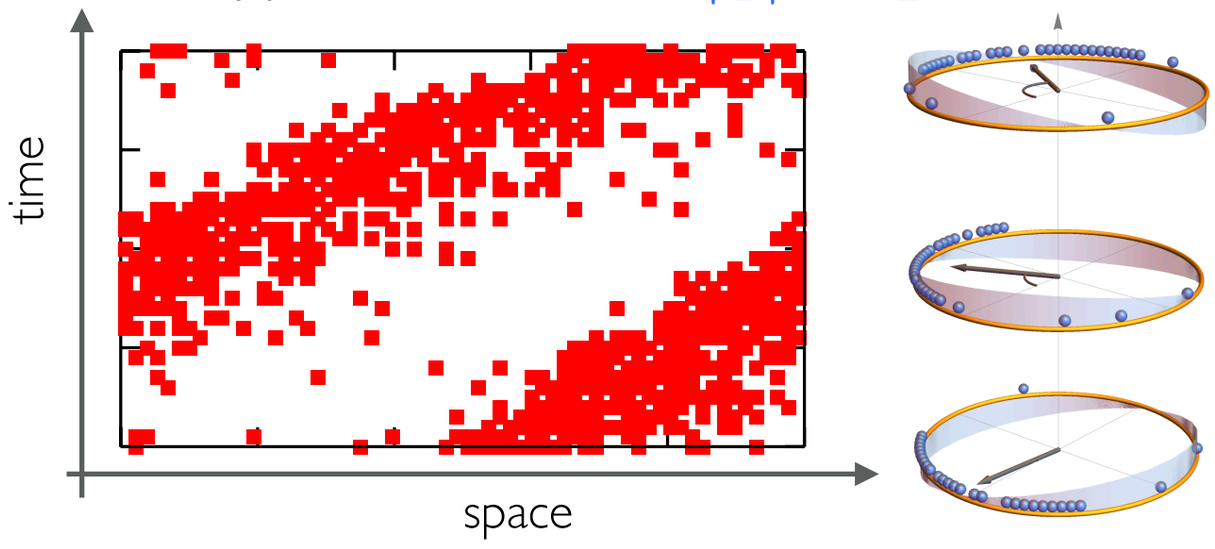
IS THIS A TIME CRYSTAL? [Wilczek, Shapere PRL 2012]

For atypical currents $|q| < q_c$



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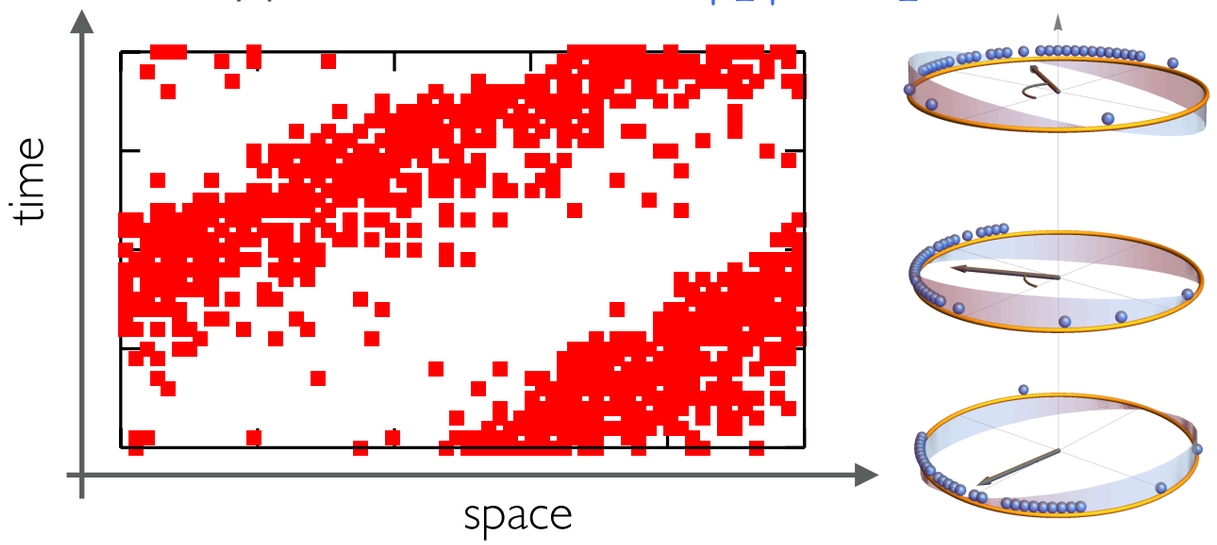
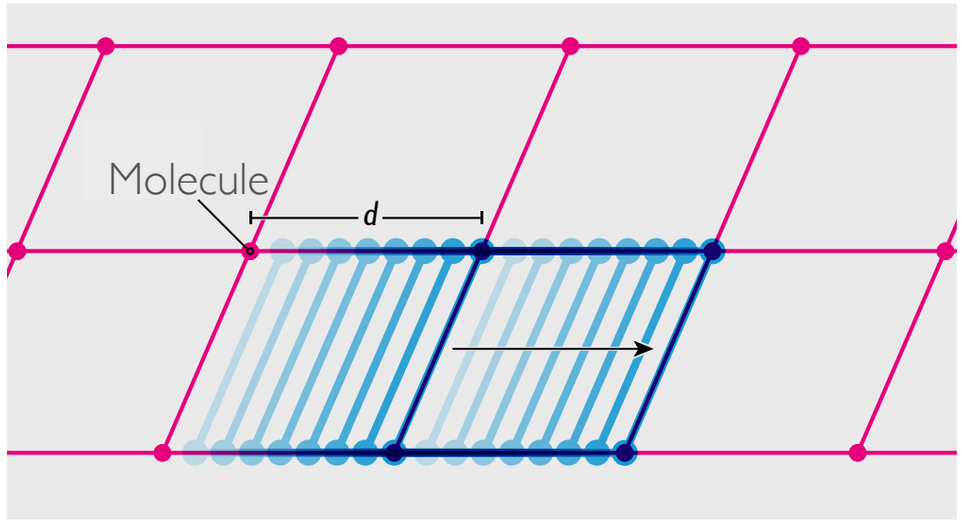


SPACE CRYSTAL

For low temperatures

[Wilczek Sci. Am. 2019]



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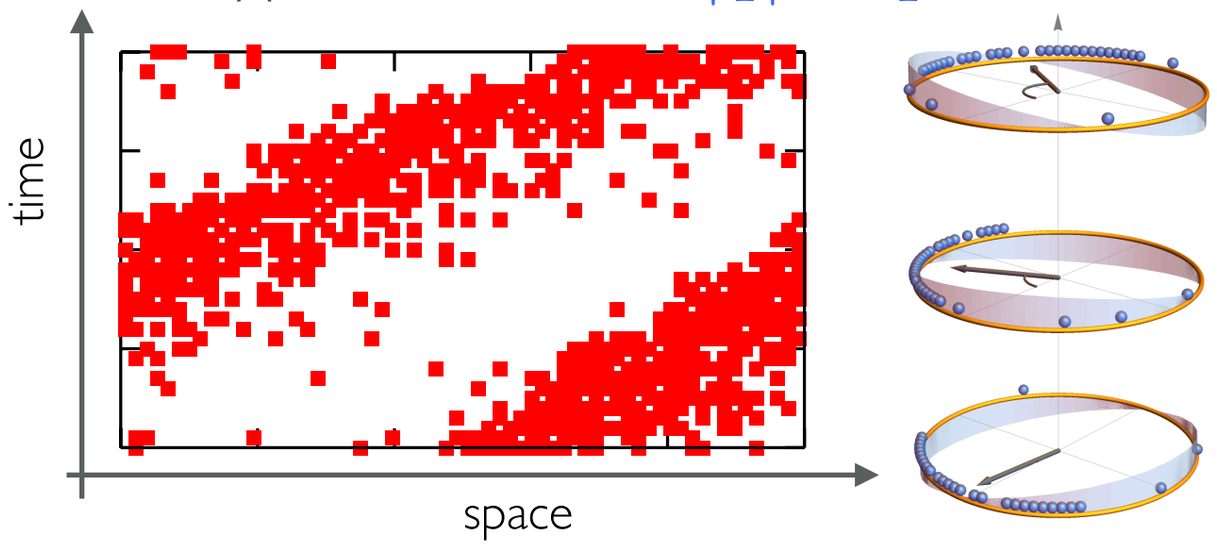

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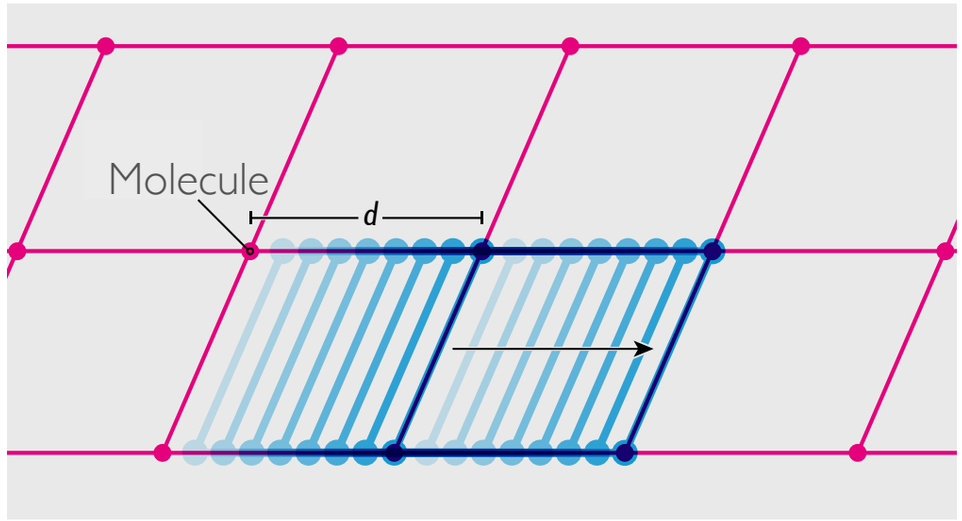

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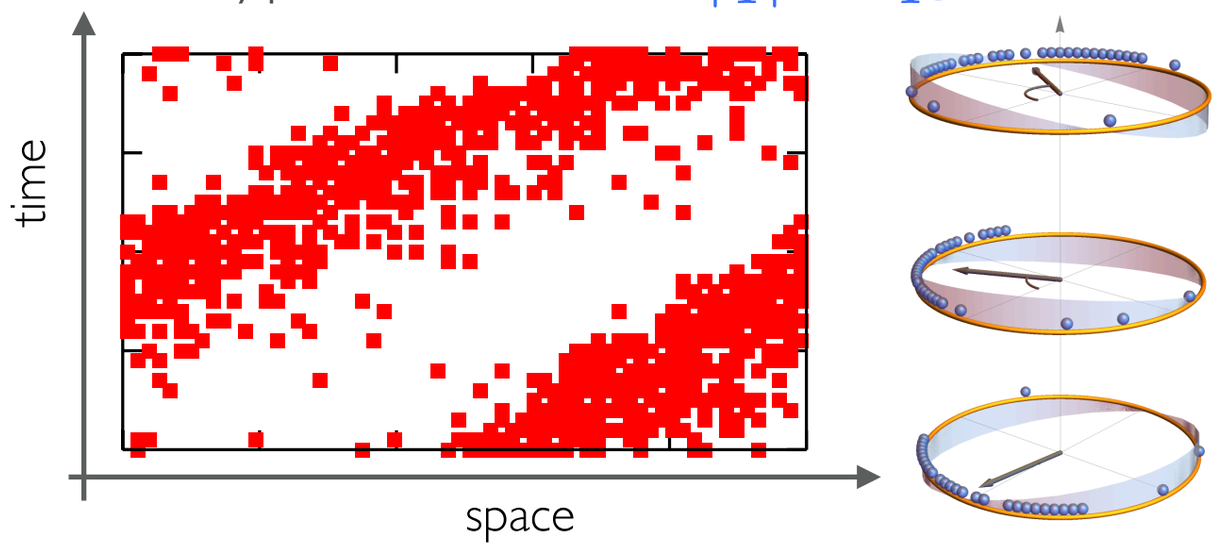
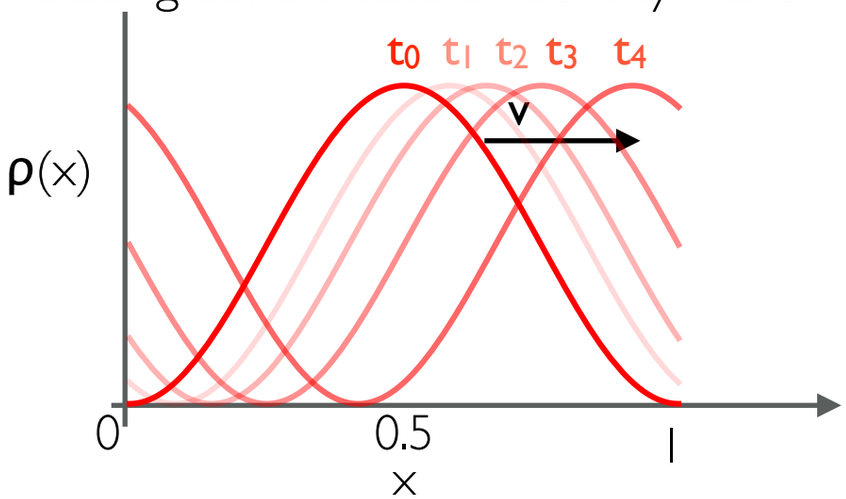


rotating condensate or density wave



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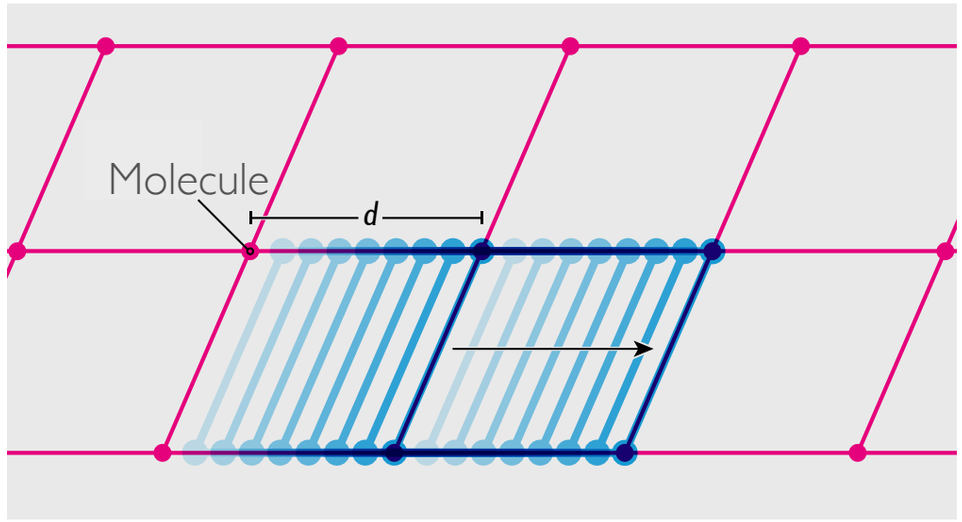

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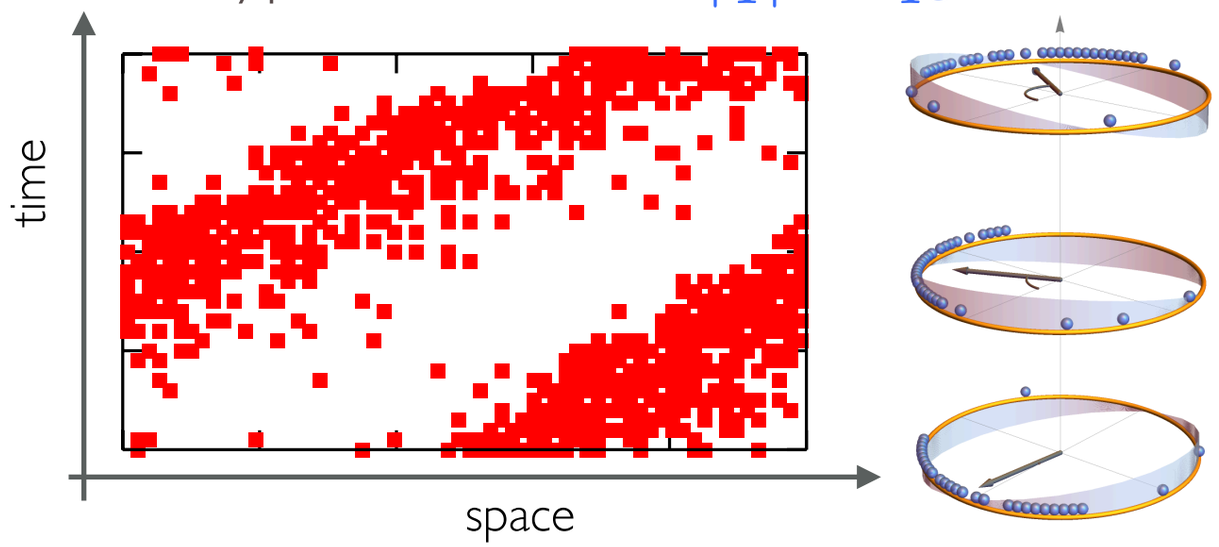

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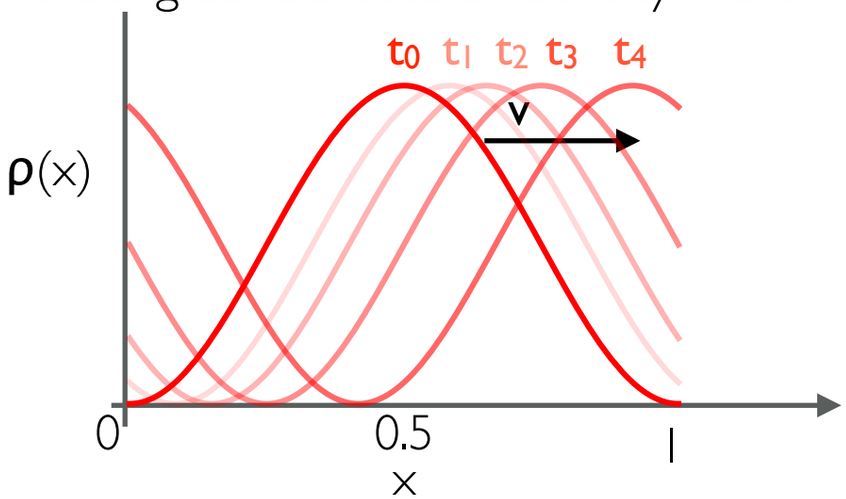

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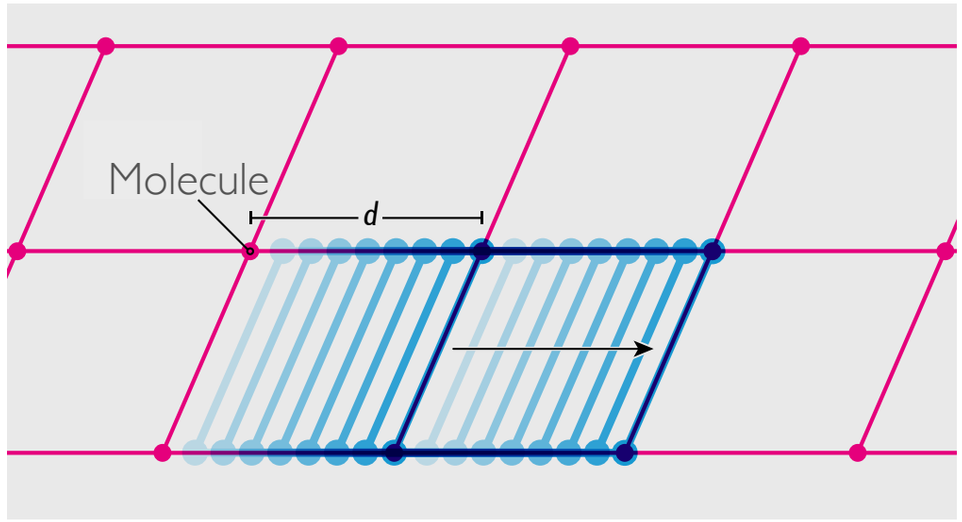

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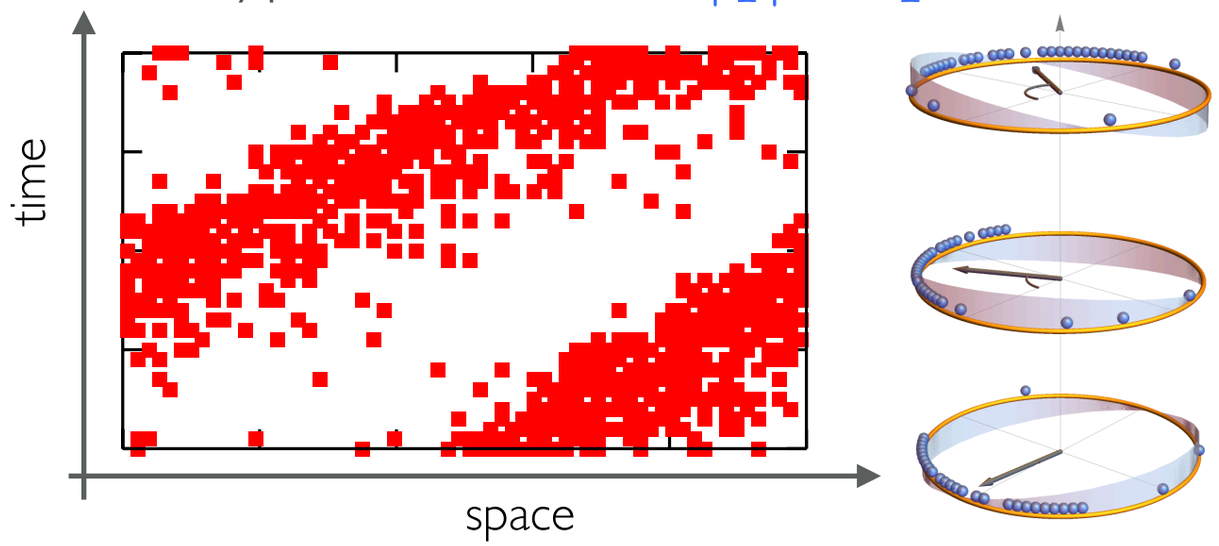

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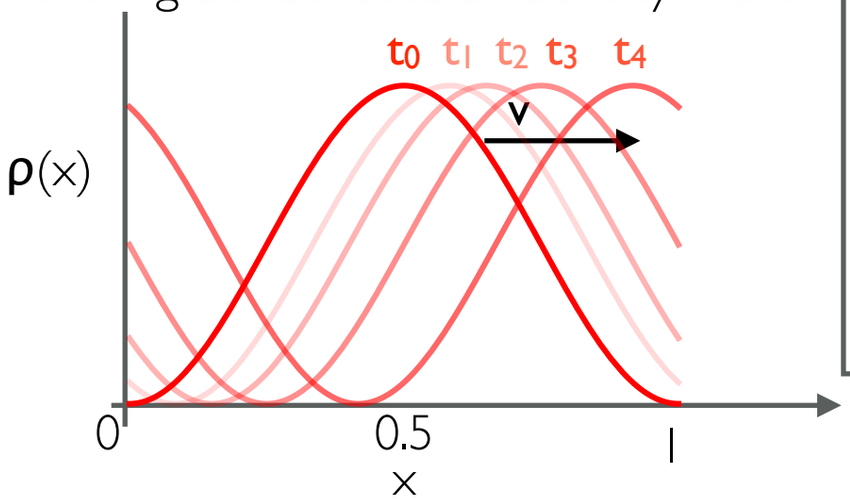

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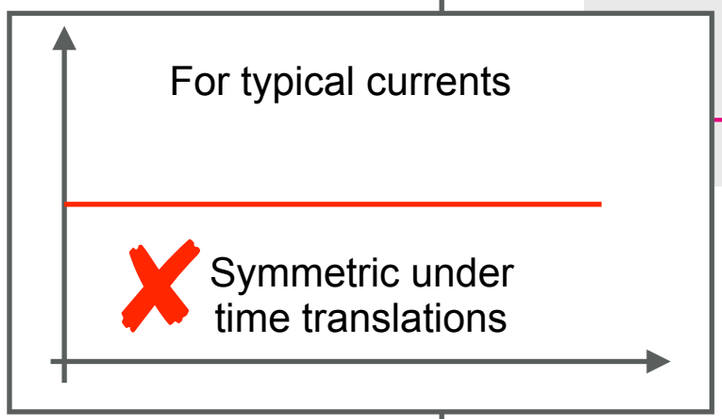

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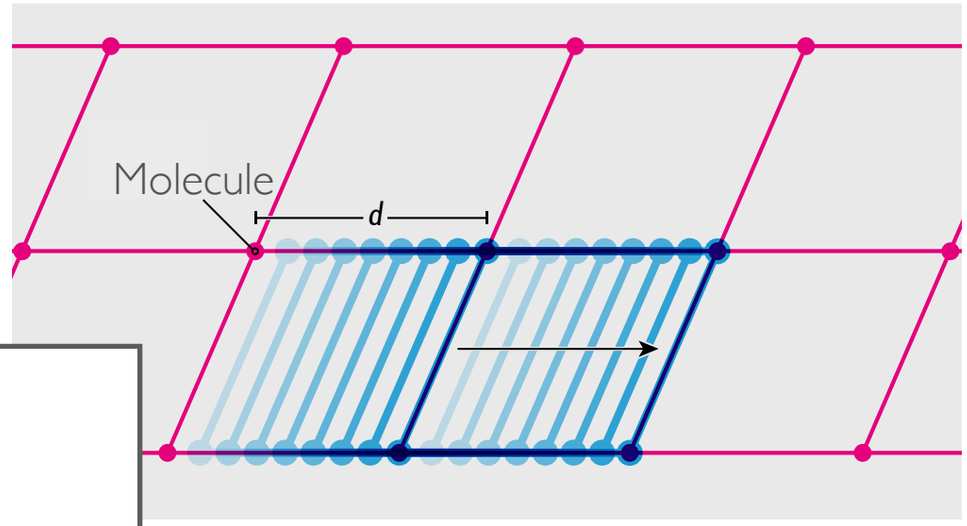

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$$\partial_t P(C,t) = \sum_{C' \neq C} W_{C' \to C} P(C',t) - R(C) P(C,t) \qquad \text{with } R(C) = \sum_{C'} W_{C \to C'}$$

- **Quantum hamiltonian formalism:** write master equation in operator form

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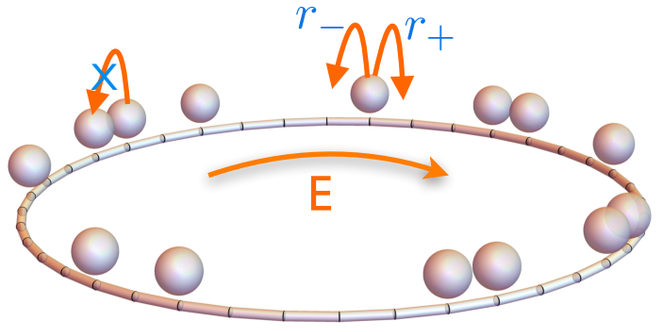
- For WASEP $|C\rangle = \bigotimes_{k=1}^{L} \begin{pmatrix} n_k \\ 1-n_k \end{pmatrix}$ with $n_k = 0,\,1$



$$\mathbb{W} = \sum_{k=1}^{L} [r_+ \hat{\sigma}^+_{k+1} \hat{\sigma}^-_k + r_- \hat{\sigma}^+_k \hat{\sigma}^-_{k+1} - r_+ \hat{n}_k (\mathbb{I} - \hat{n}_{k+1}) - r_- \hat{n}_{k+1} (\mathbb{I} - \hat{n}_k)]$$

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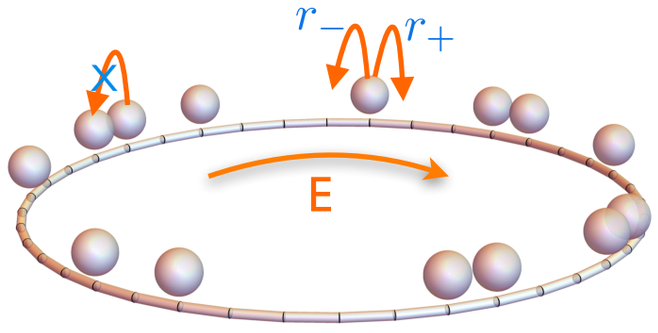
- For **WASEP** $|C\rangle = \bigotimes_{k=1}^{L} \begin{pmatrix} n_k \\ 1-n_k \end{pmatrix}$ with $n_k = 0,\, 1$



$$\mathbb{W} = \sum_{k=1}^{L} \Big[\underbrace{r_+ \hat{\sigma}^+_{k+1} \hat{\sigma}^-_k}_{\text{Jumps to the right}} + \underbrace{r_- \hat{\sigma}^+_k \hat{\sigma}^-_{k+1}}_{\text{Jumps to the left}} - \underbrace{r_+ \hat{n}_k (\mathbb{I} - \hat{n}_{k+1}) - r_- \hat{n}_{k+1} (\mathbb{I} - \hat{n}_k)}_{\text{Conservation of probability}} \Big]$$

CURRENT FLUCTUATIONS FROM MICROSCOPICS

- **Ensemble of trajectories** conditioned on Q:

[Ruelle, Gartner&Ellis, Lebowitz&Spohn, Lecomte et al, and many others]

$$P_t(Q) \sim \mathrm{e}^{-tG(Q/t)}$$

- Dynamical partition function:

$$Z_t(\lambda) = \sum_Q P_t(Q)\mathrm{e}^{\lambda Q} \sim \mathrm{e}^{t\theta(\lambda)} \qquad \theta(\lambda) = -\min_q [G(q) - \lambda q]$$

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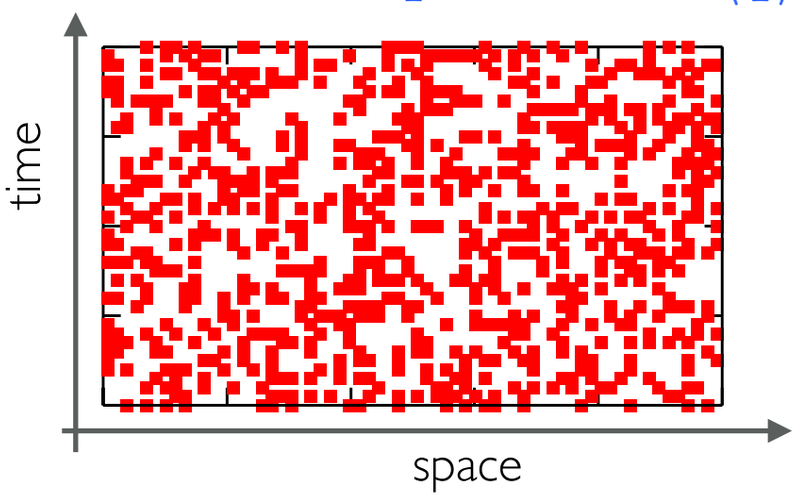

$\mathbb{W}^{\lambda}$ spectrum changes radically across the DPT

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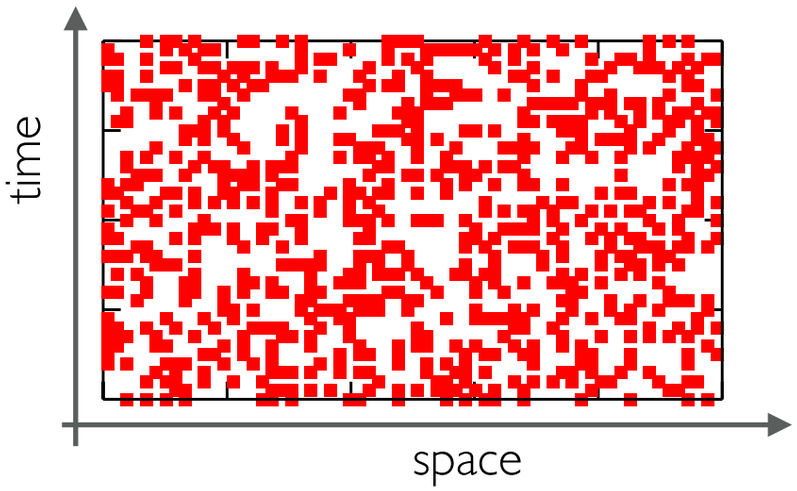

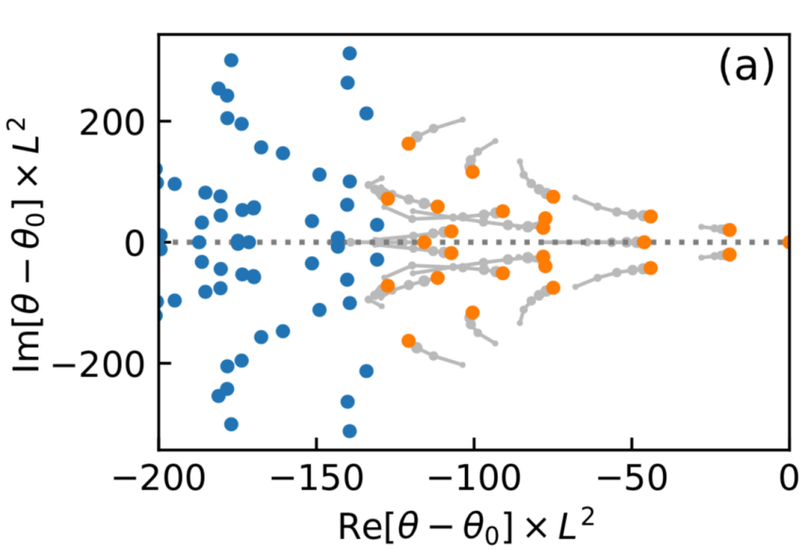

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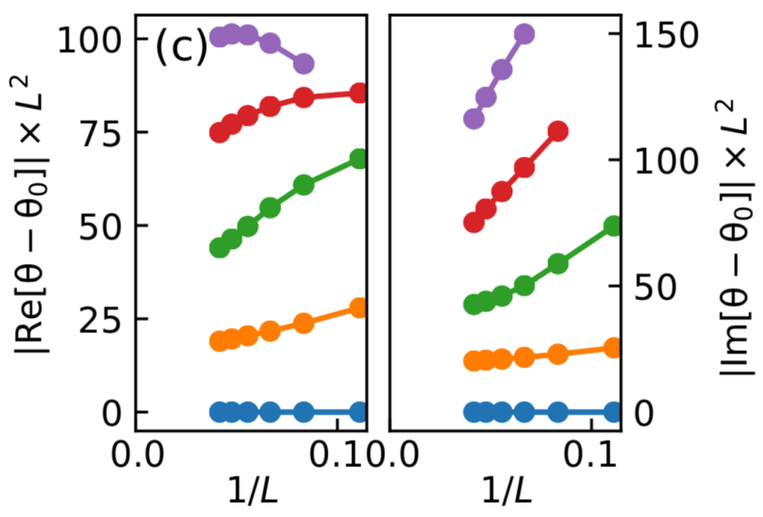

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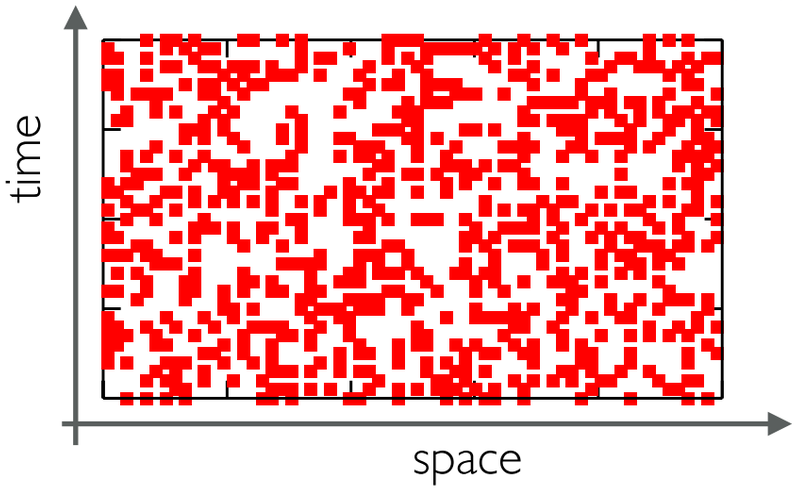

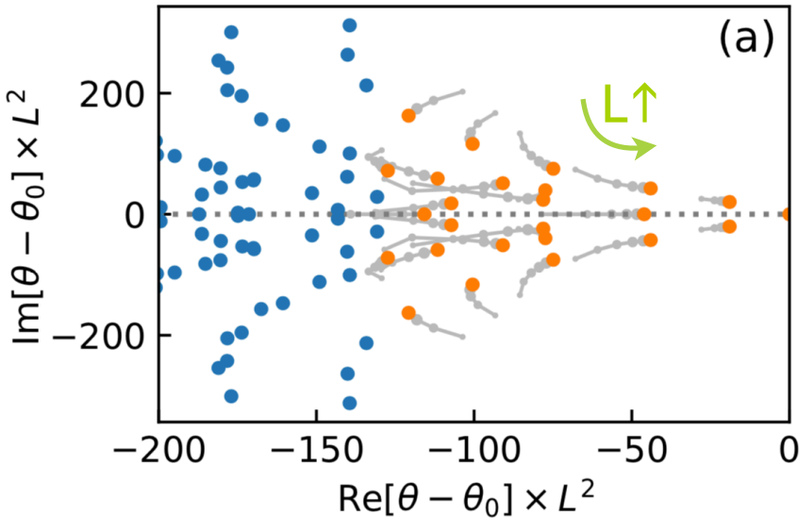

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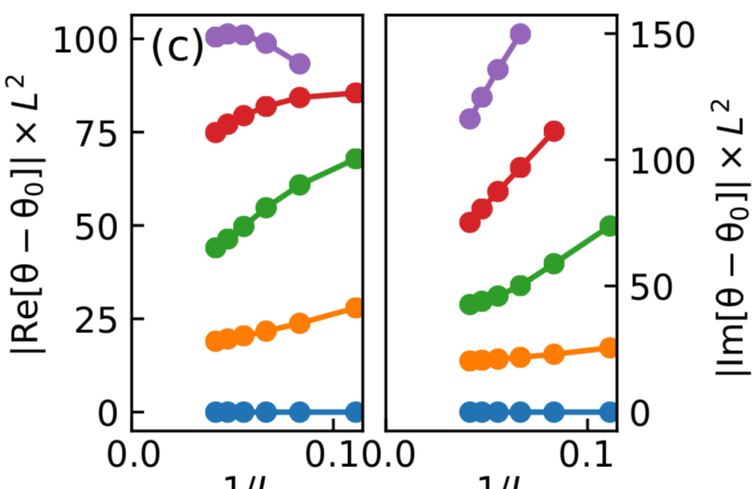

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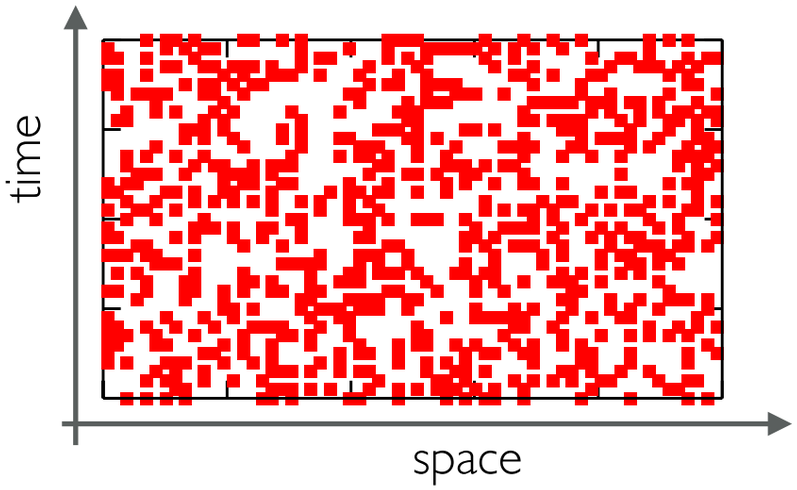

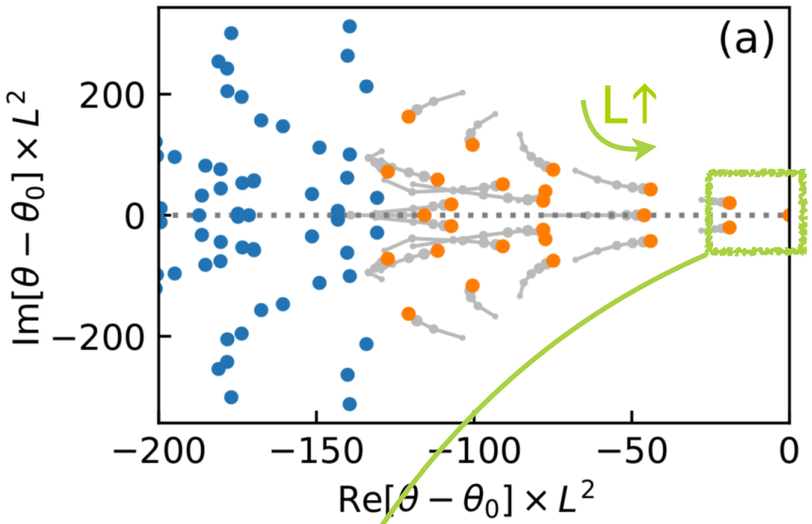

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⇓

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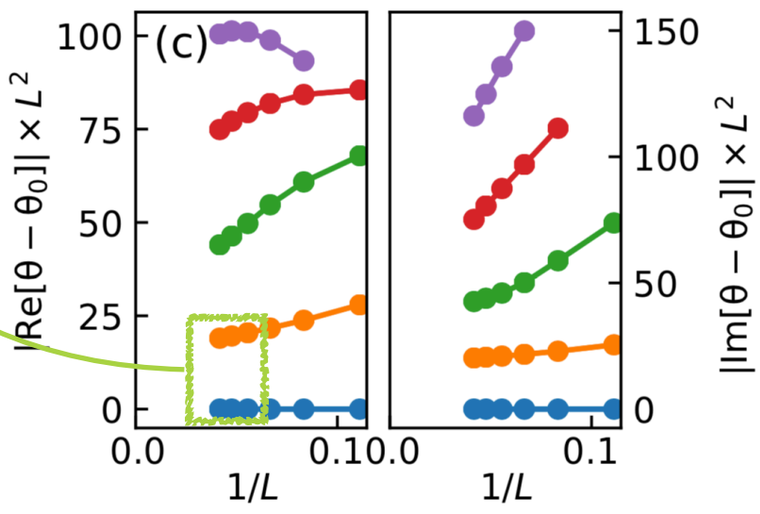

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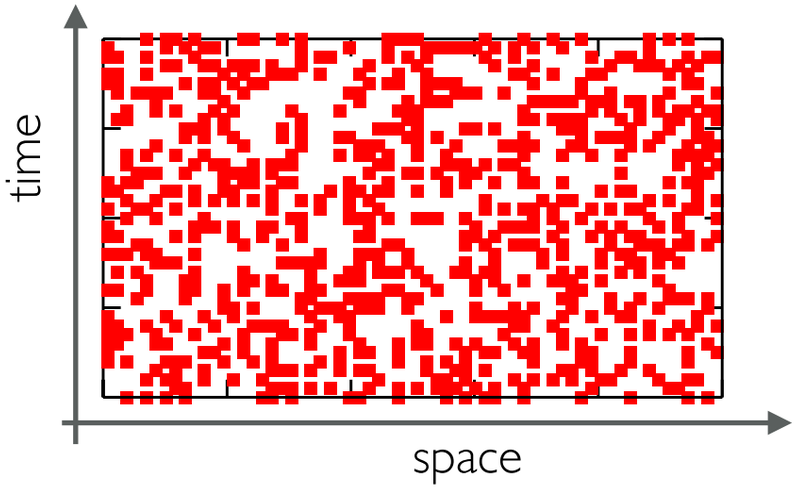

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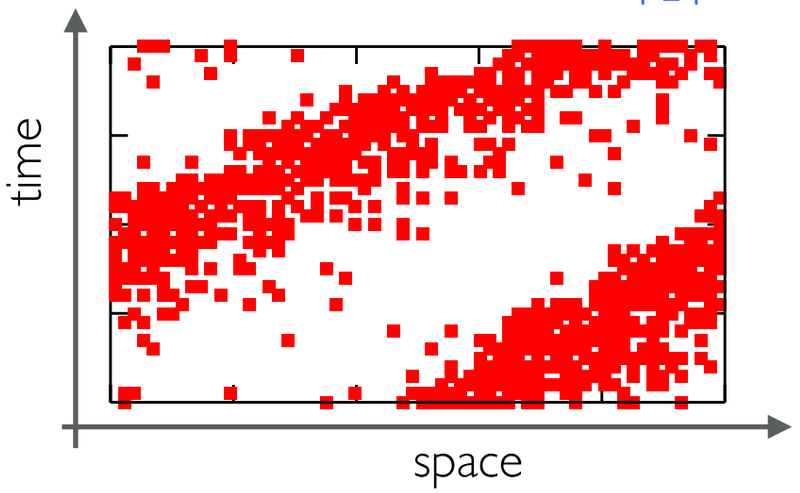

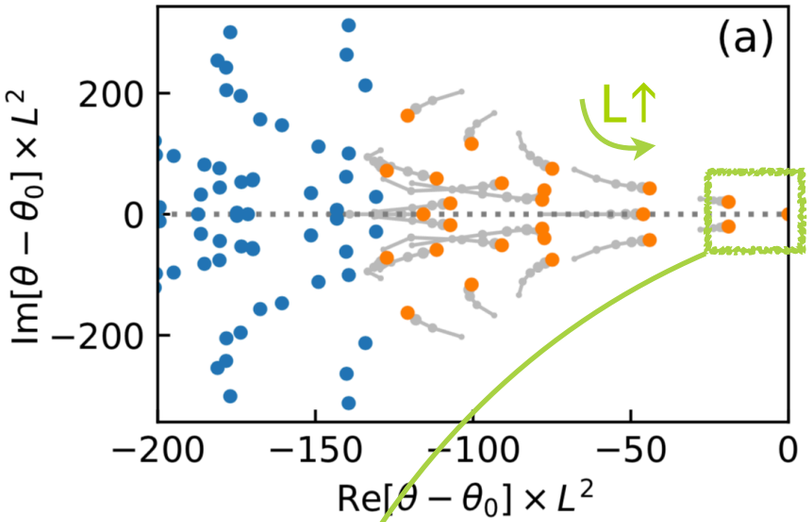

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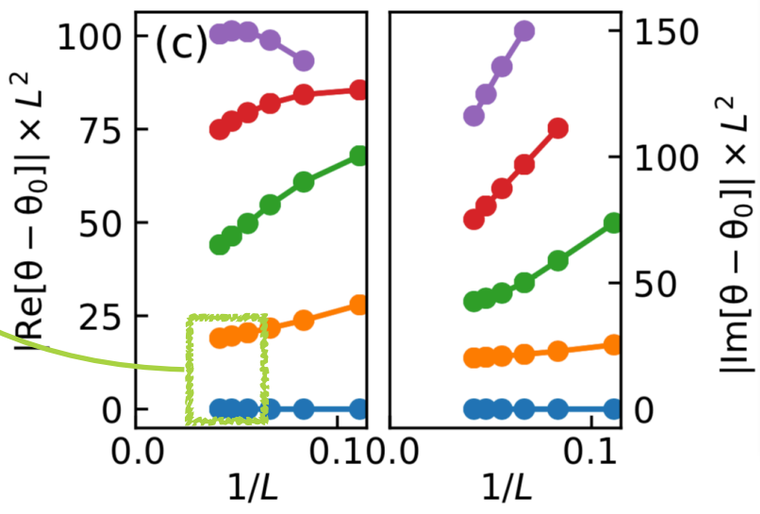

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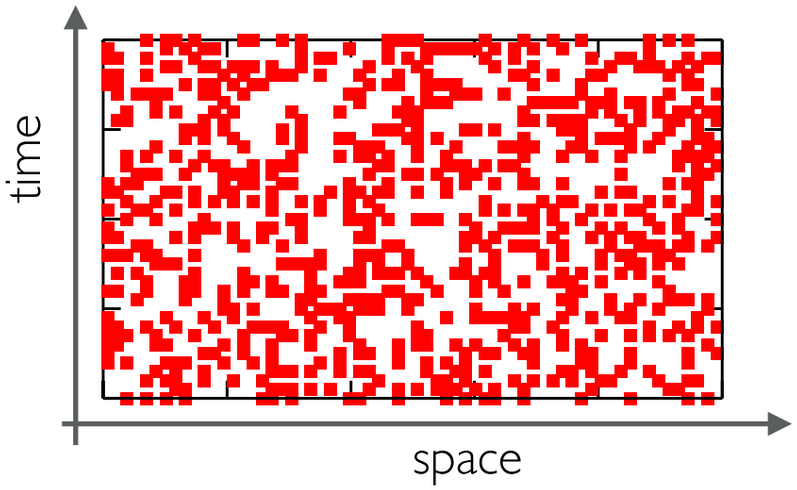

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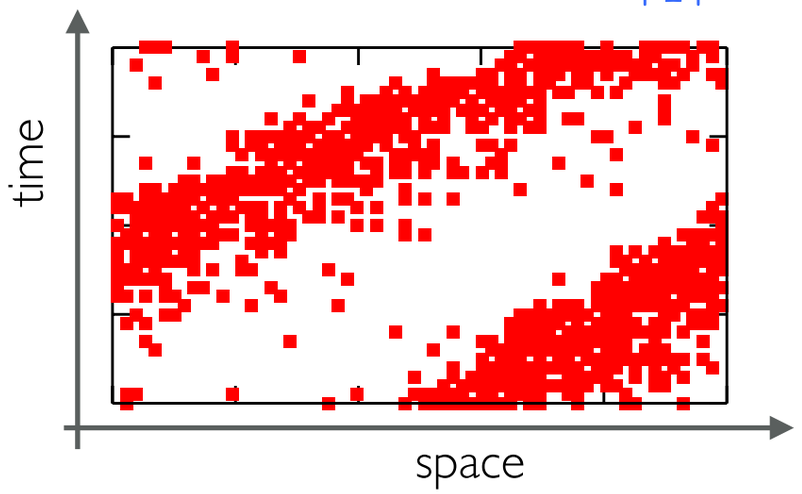

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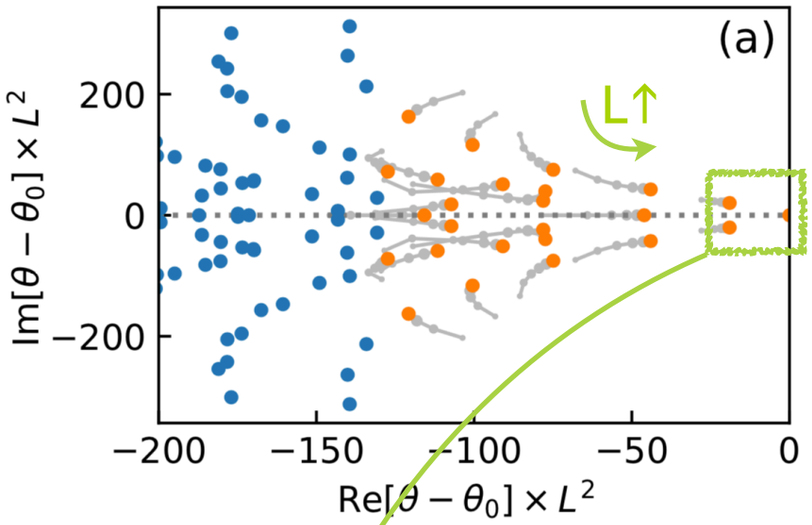

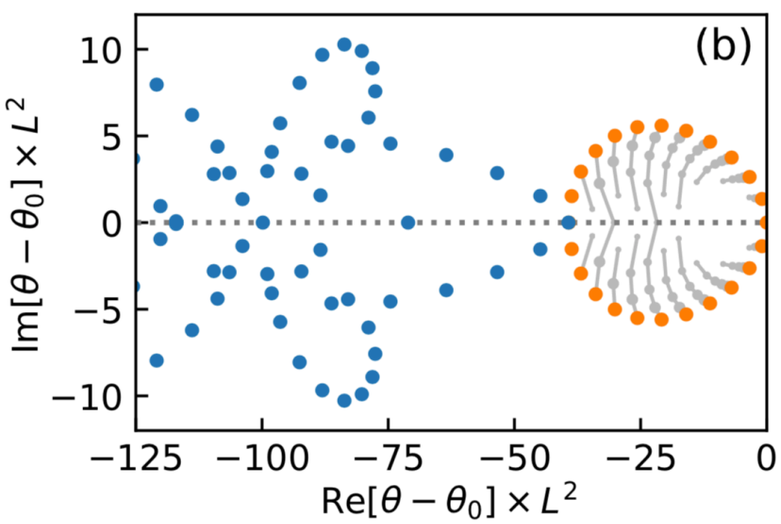

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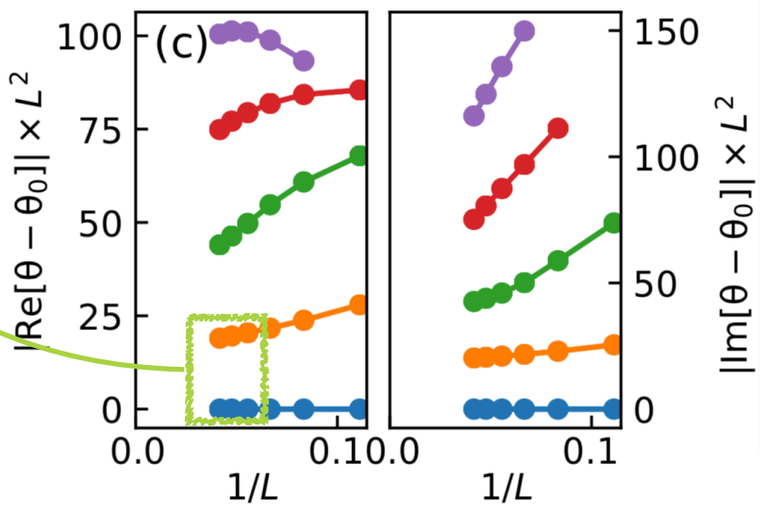

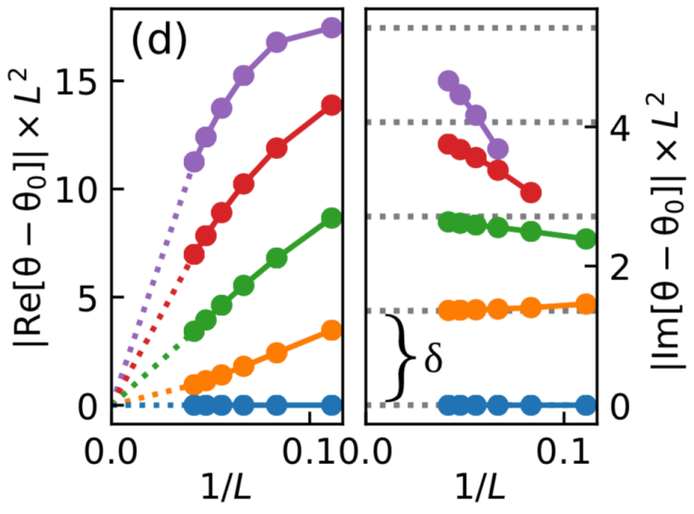

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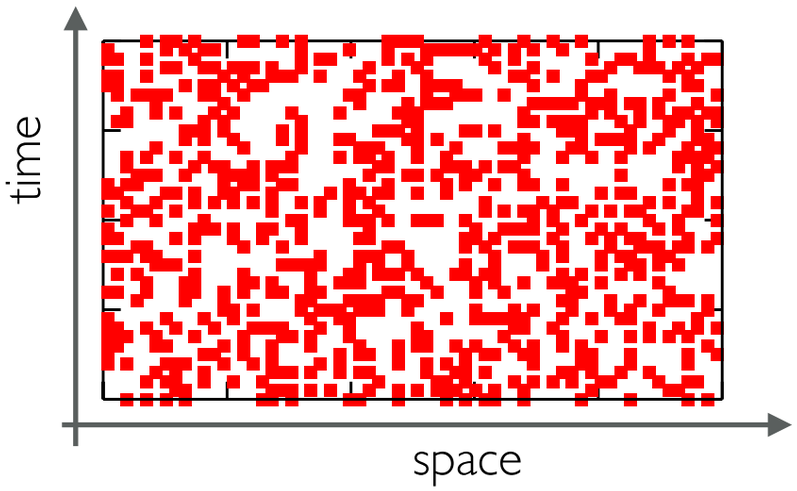

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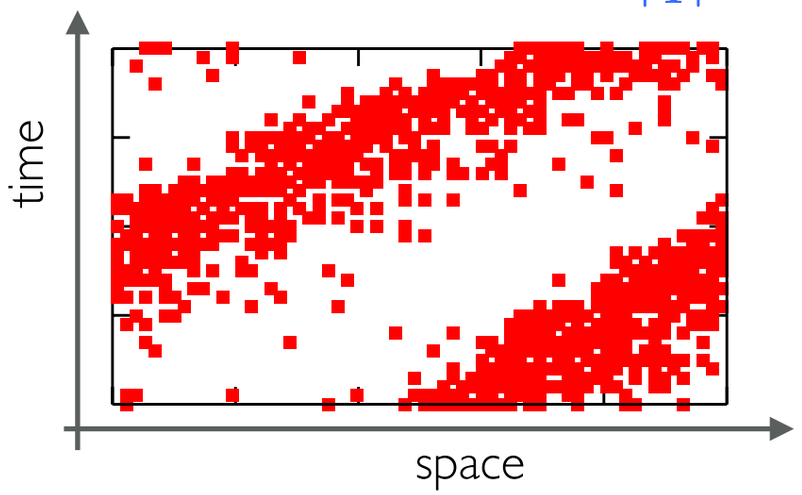

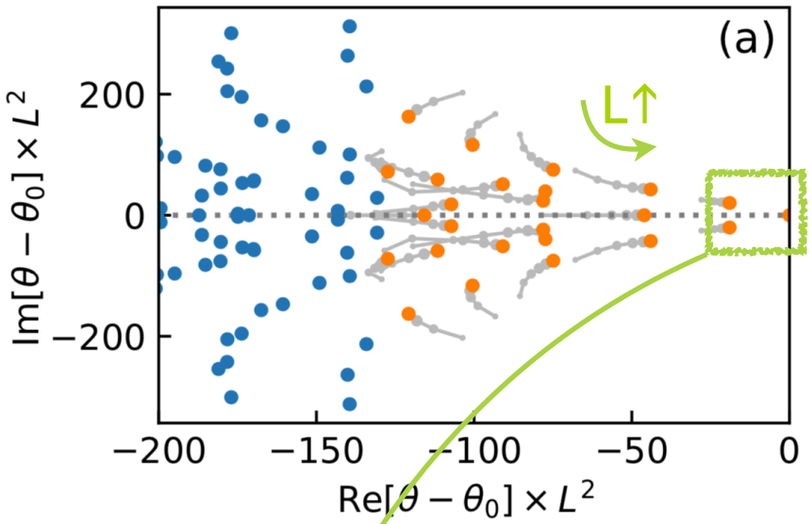

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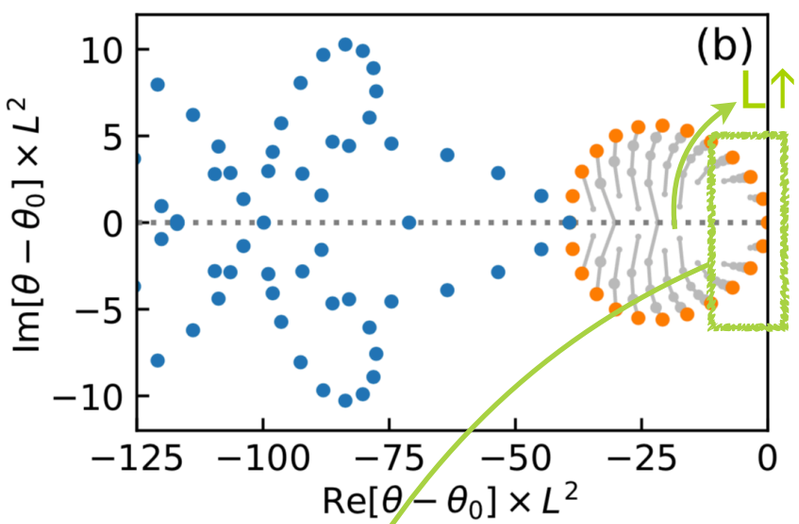

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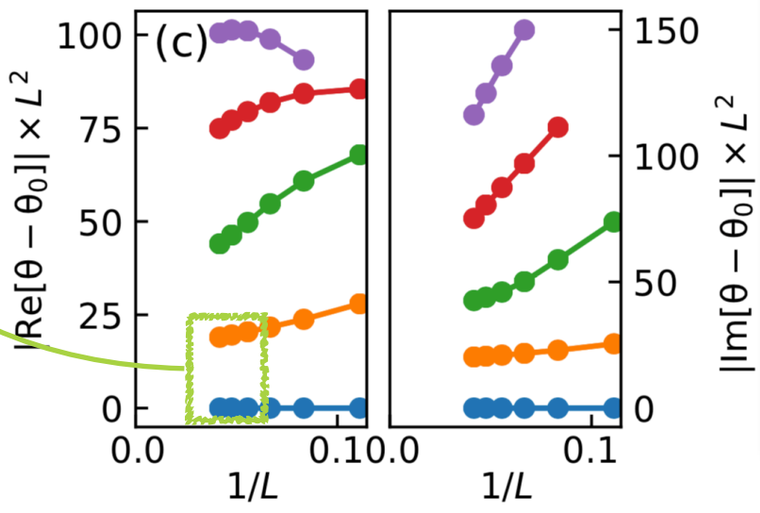
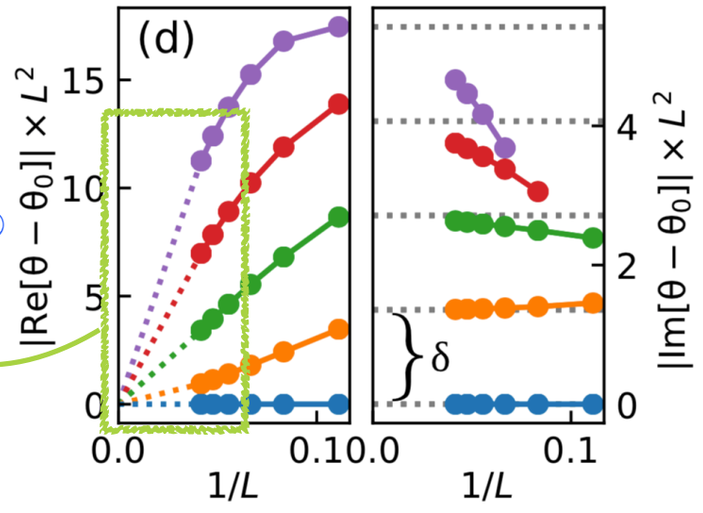


Vanishing spectral gap of $\mathbb{W}^\lambda$ as $L \to \infty$

Power-law decay as L↑



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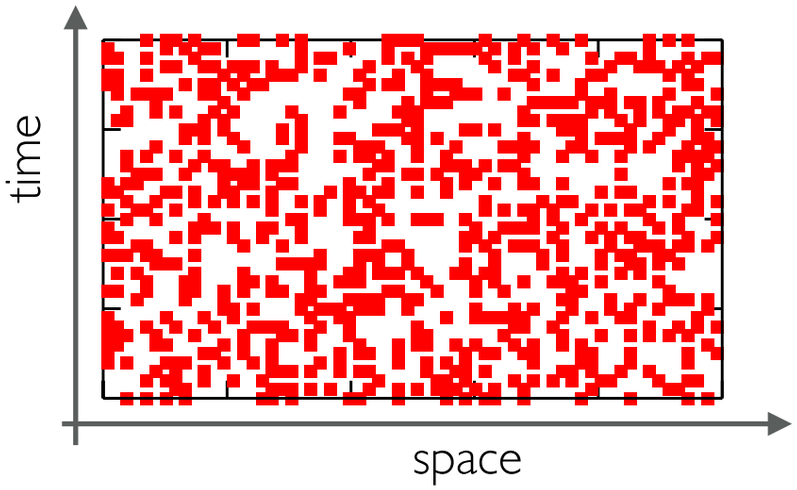

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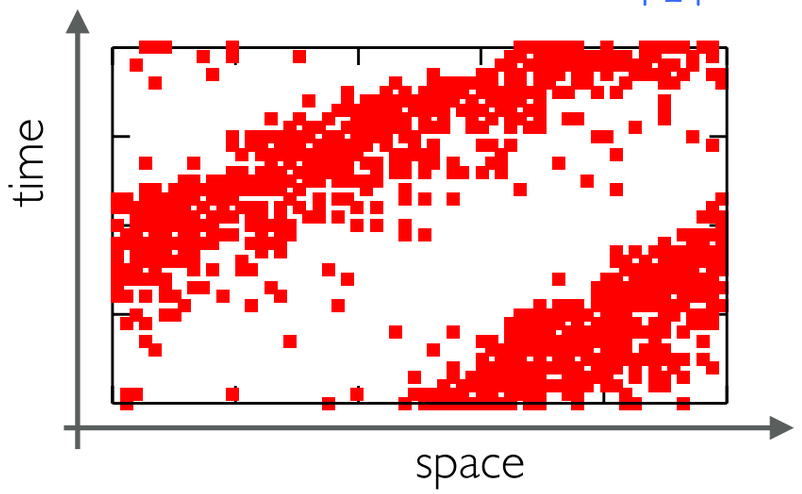

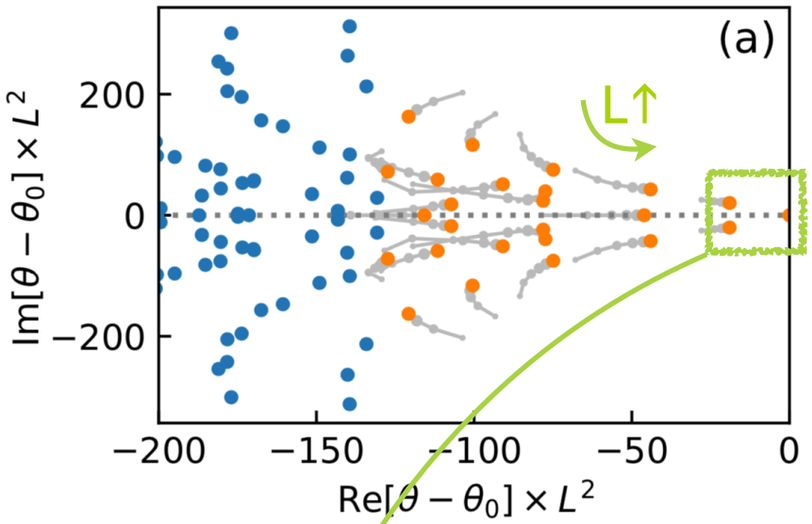

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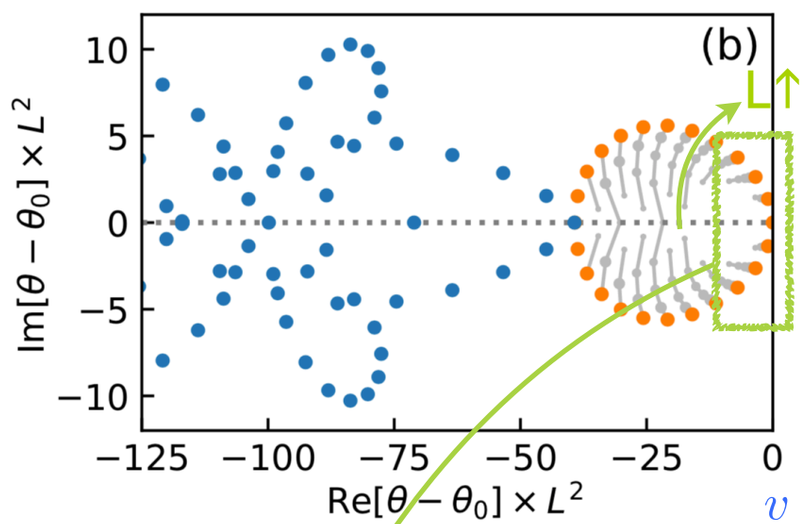

$|P(t)\rangle = |P(t+T)\rangle$

$T = \frac{2\pi}{\delta}$

Band structure in **imaginary part** of gap-closing eigenvals

$v = L/T = L\delta/2\pi$

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⇓

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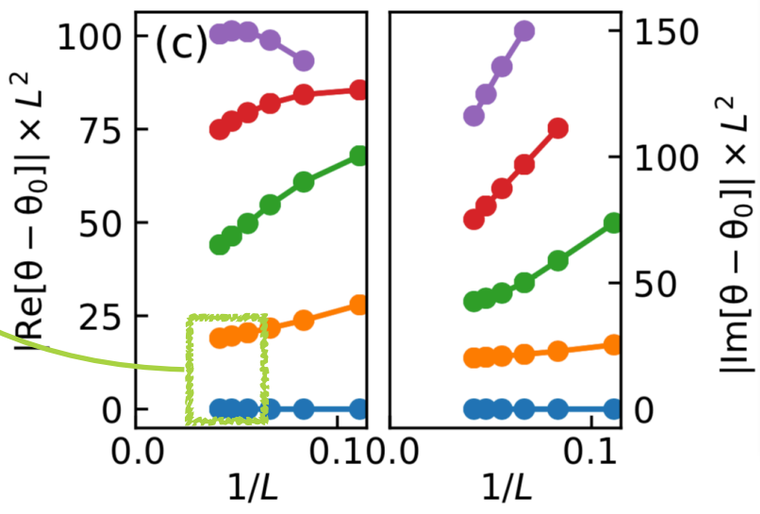

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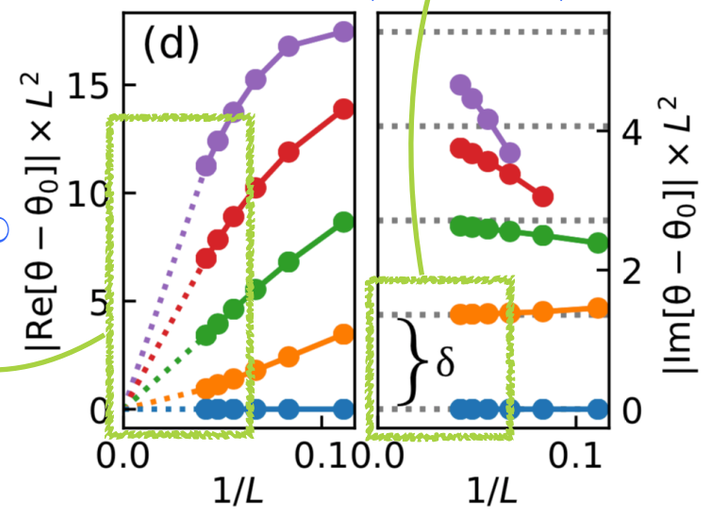

MAKING RARE EVENTS TYPICAL

- We have a **time crystal for rare current fluctuations** … WHO CARES?
- $\mathbb{W}^\lambda$ generates **atypical trajectories** but $\sum_C \langle C| \mathbb{W}^\lambda \neq 0$ **(non-physical!)**

$\mathbb{W}^\lambda_{\mathrm{D}}$

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$$\mathbb{W}_{\mathrm{D}}^\lambda \equiv \mathbb{L}_0 \mathbb{W}^\lambda \mathbb{L}_0^{-1} - \theta_0(\lambda) \qquad \text{with } (\mathbb{L}_0)_{ij} = (\langle L_0^\lambda|)_i \delta_{ij}$$

- $\mathbb{W}_{\mathrm{D}}^\lambda$ now **conserves probability** (physical!)

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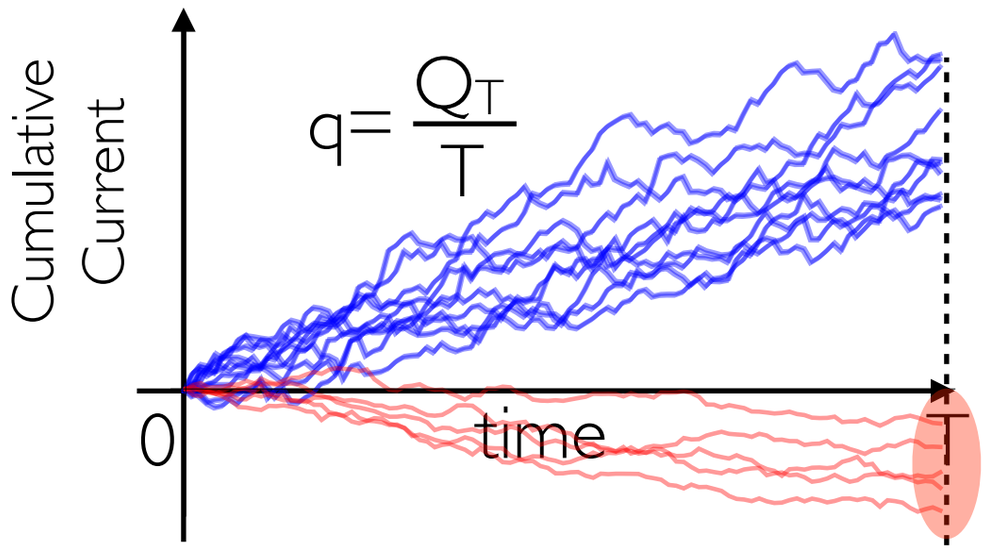

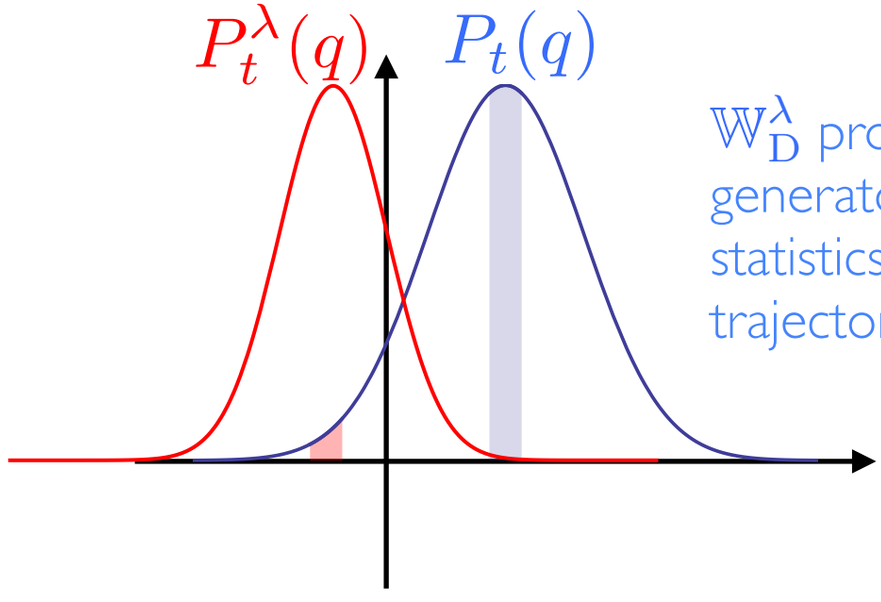

$\mathbb{W}_{\mathrm{D}}^\lambda$ proper stochastic generator for the statistics of atypical trajectories

DOOB'S SMART FIELD

- Write **Doob's dynamics** in terms of original **WASEP dynamics + smart field** E_λ^D

$$(\mathbb{W}_D^\lambda)_{C\to C'} = \mathbb{W}_{C\to C'} \mathrm{e}^{\pm E_\lambda^D/L} \qquad \Rightarrow \qquad (E_\lambda^D)_{C\to C'} = \lambda \pm L \ln\left(\frac{\langle L_0^\lambda | C'\rangle}{\langle L_0^\lambda | C\rangle}\right)$$

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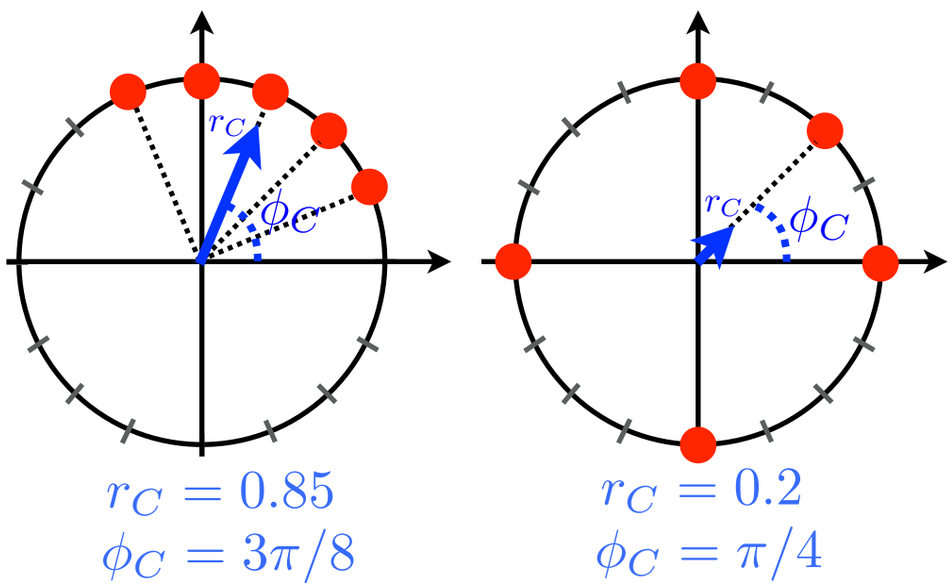
- Write **Doob's dynamics** in terms of original **WASEP dynamics + smart field** E^D_λ

$$(\mathbb{W}^\lambda_D)_{C\to C'} = \mathbb{W}_{C\to C'} \mathrm{e}^{\pm E^D_\lambda / L} \qquad \Rightarrow \qquad (E^D_\lambda)_{C\to C'} = \lambda \pm L \ln \left(\frac{\langle L^\lambda_0 | \, C' \rangle}{\langle L^\lambda_0 | \, C \rangle} \right)$$

- Study dependence of E^D_λ on the **packing order parameter** $r_C \equiv |z_C|$

$$z_C = N^{-1} \sum_{k=1}^{L} n_k(C) \mathrm{e}^{i2\pi k/L} = r_c \mathrm{e}^{i\phi_C}$$



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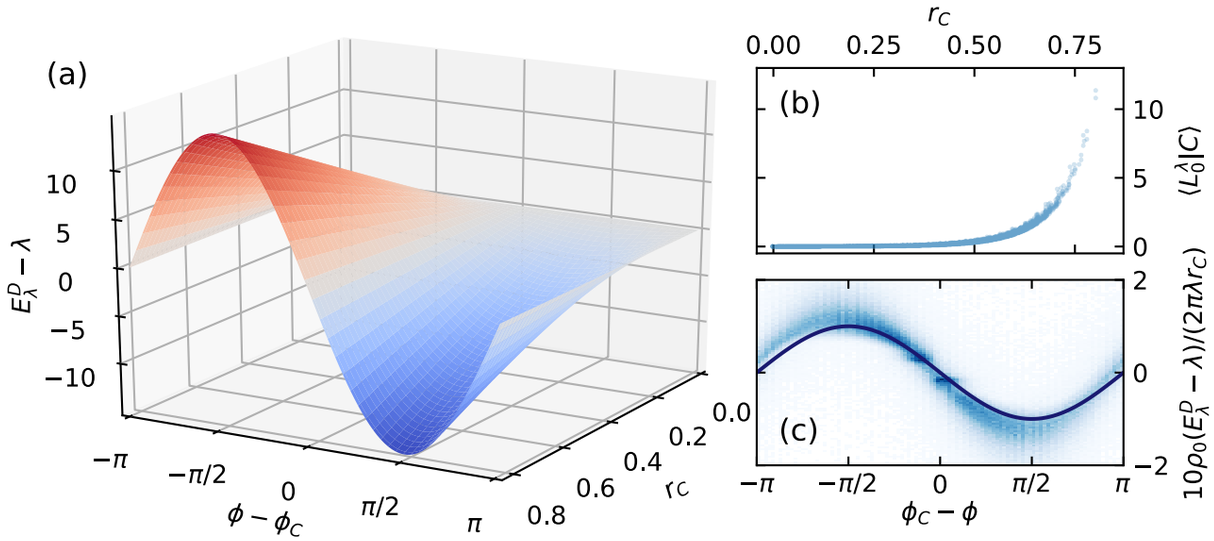

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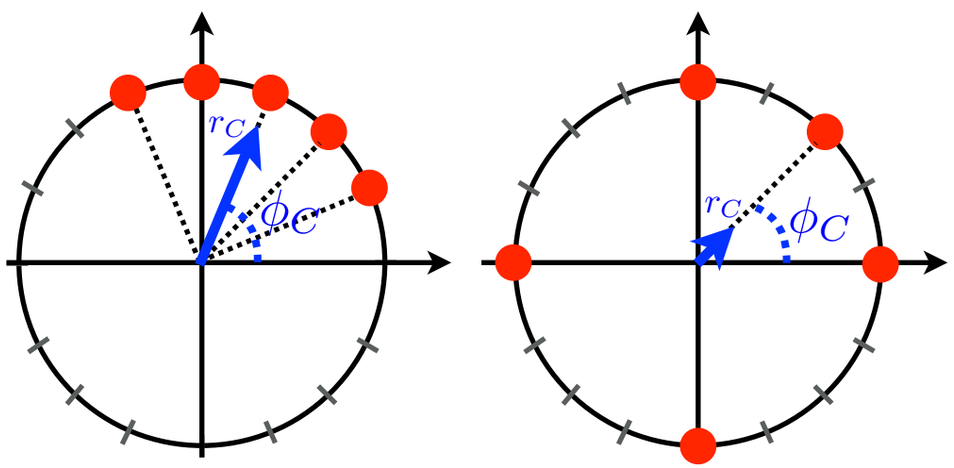

$r_C = 0.85$
$\phi_C = 3\pi/8$

$r_C = 0.2$
$\phi_C = \pi/4$

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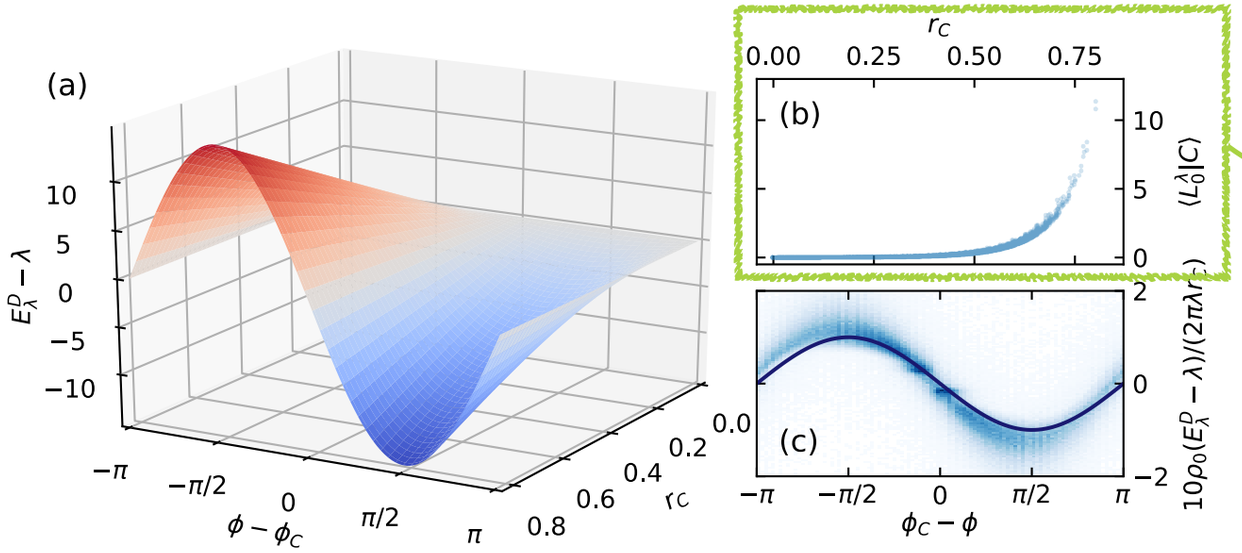

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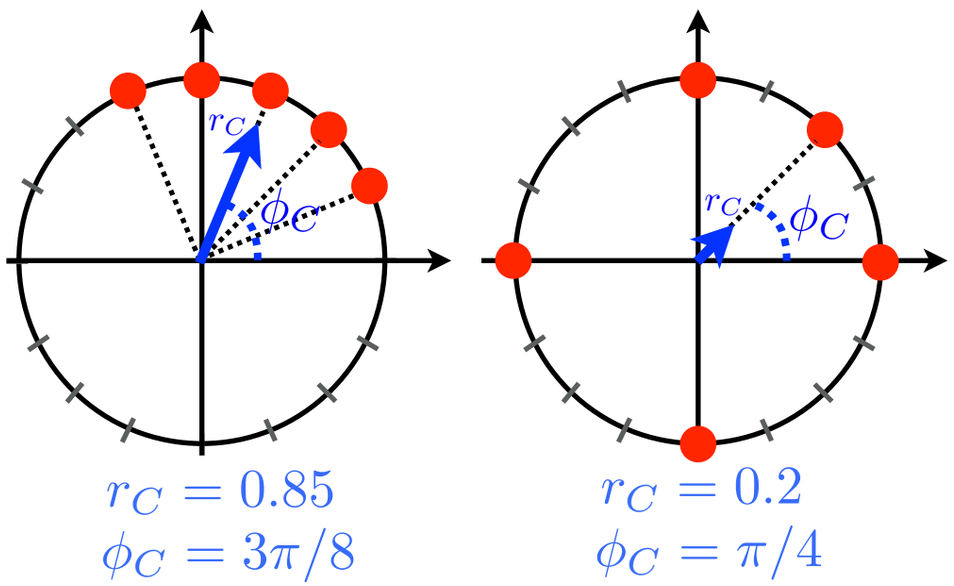
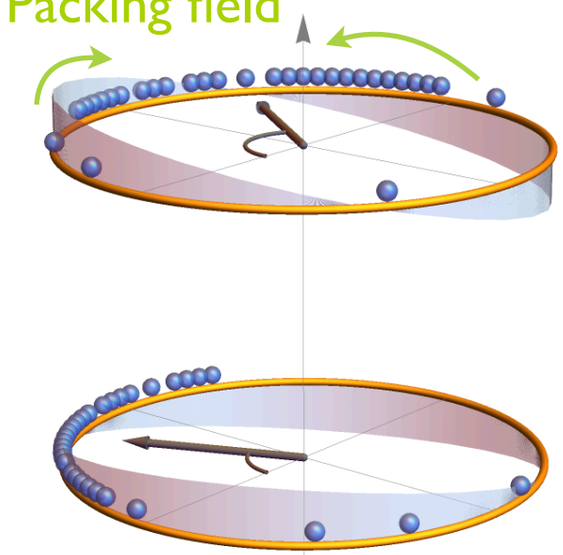
$$\langle L_0^\lambda | C\rangle \simeq f_{\lambda,L}(r_C)$$

$$\Downarrow$$

$$E_\lambda^D \approx \lambda + \frac{\pi}{5\rho_0}\lambda r_C \sin(\phi_k - \phi_C)$$

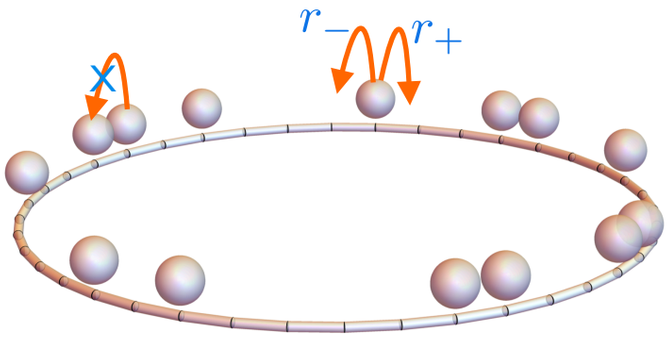


Packing field



TIME CRYSTAL LATTICE GAS (TCLG)

- Inspired by E_λ^D we now propose a **variant of WASEP** with **configuration-dependent packing field**



$$E_\lambda(C;k) = E + \lambda + 2\lambda r_C \sin(\phi_k - \phi_C)$$

$$r_\pm(C;k) = \frac{1}{2} \mathrm{e}^{\pm E_\lambda(C;k)/L}$$

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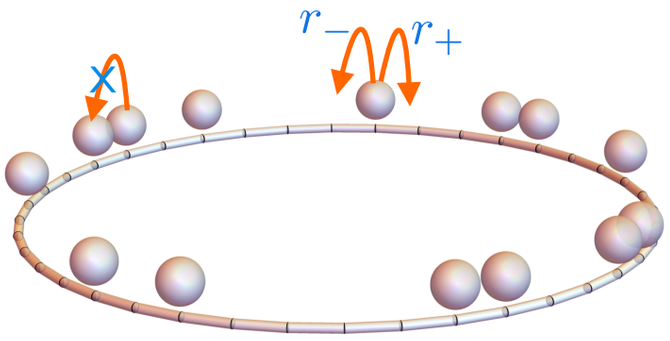

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Hydrodynamic limit

$$\partial_t \rho = -\partial_x\Big(-D\partial_x\rho + \sigma(\rho)E_\lambda(\rho;x)\Big)$$

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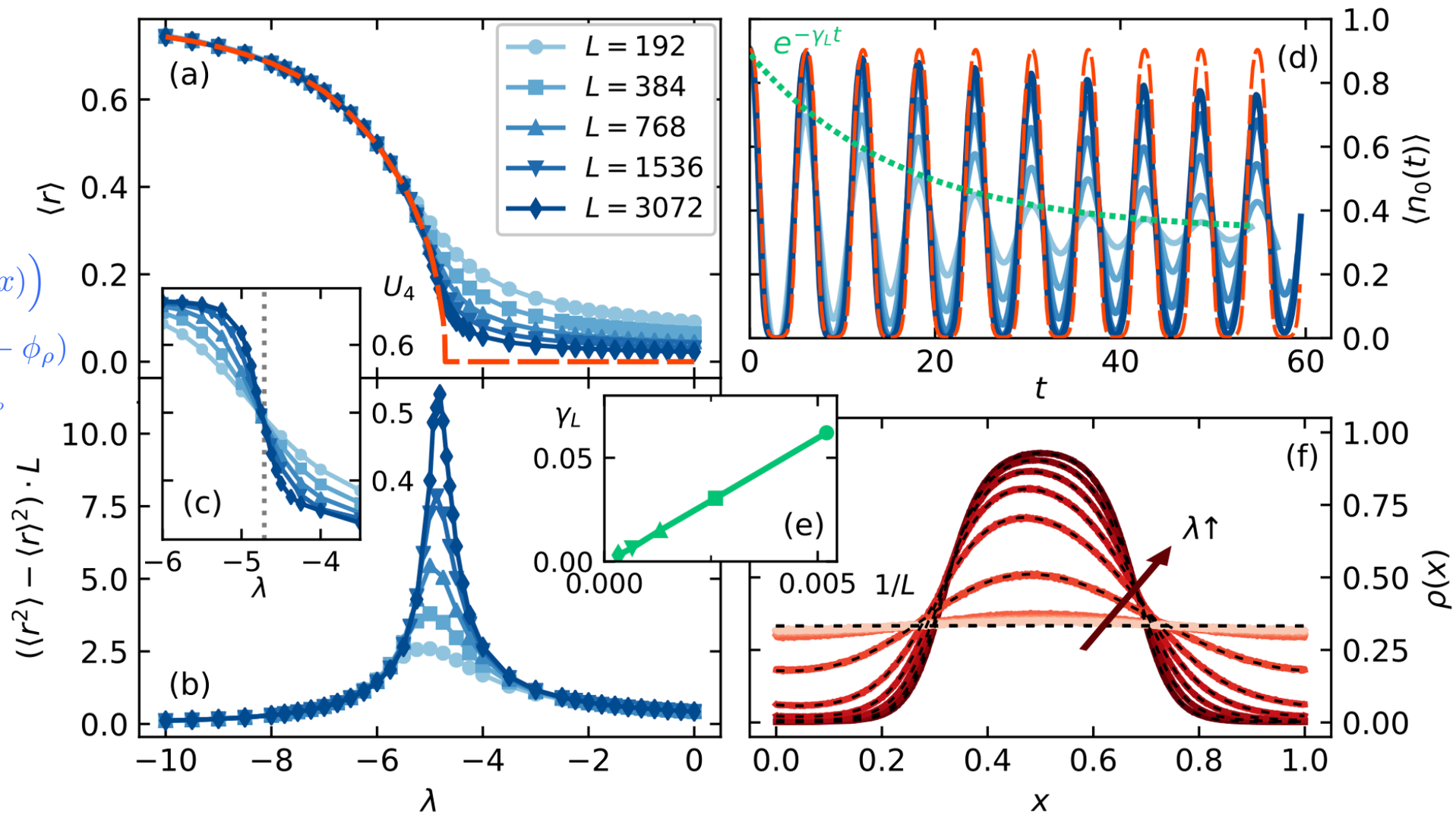

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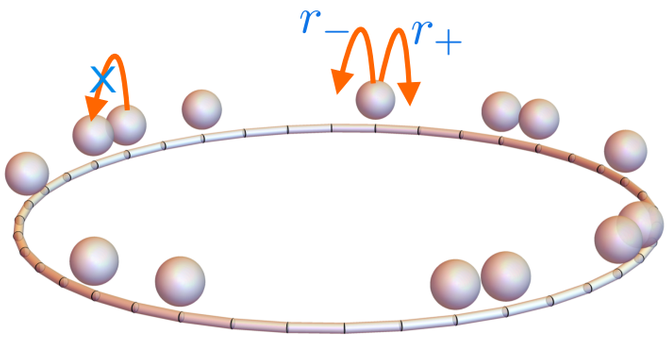

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Steady-state phase transition to time-crystal phase

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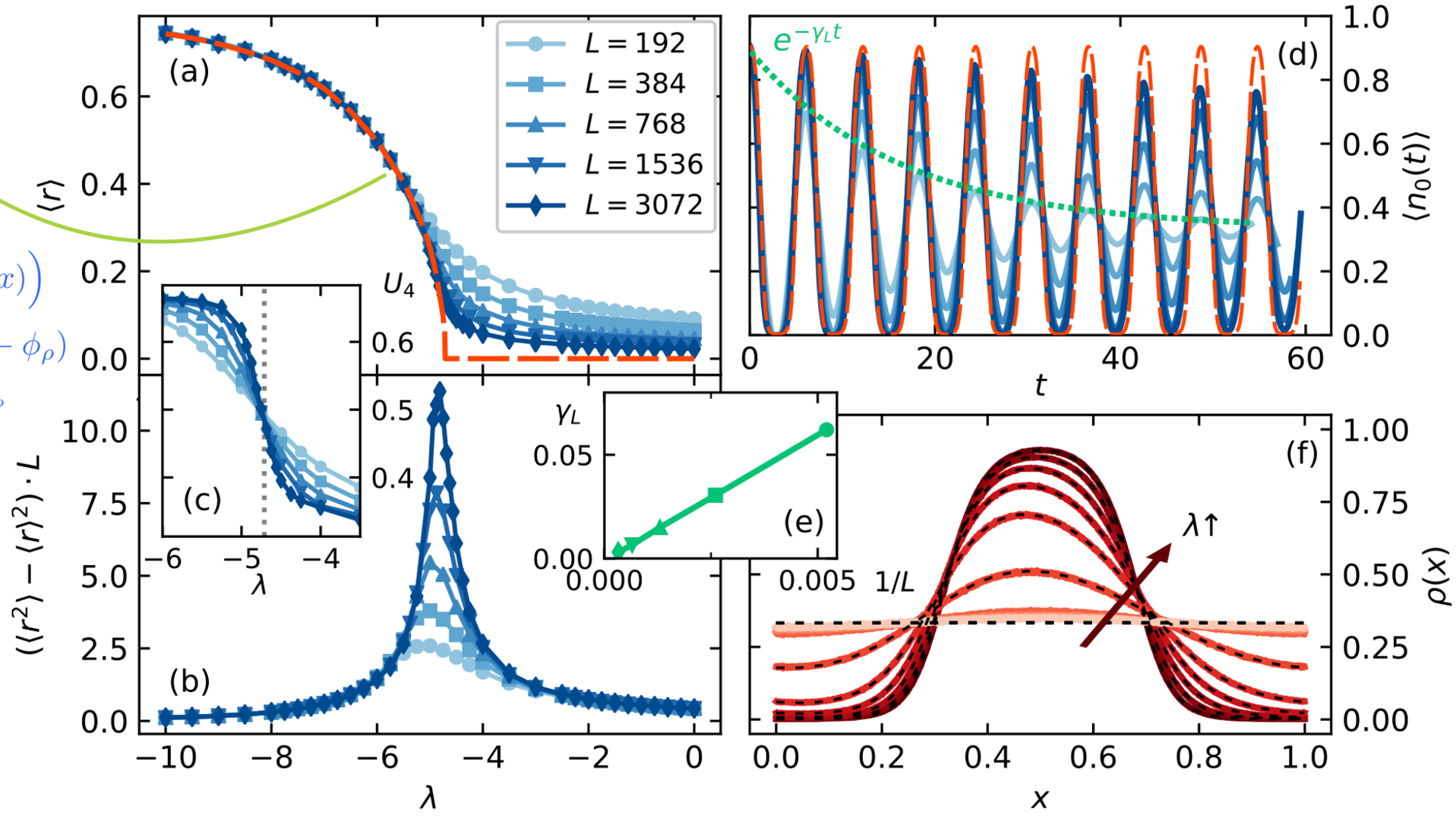

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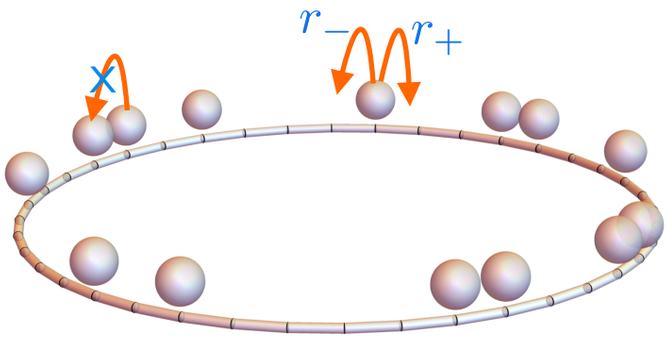

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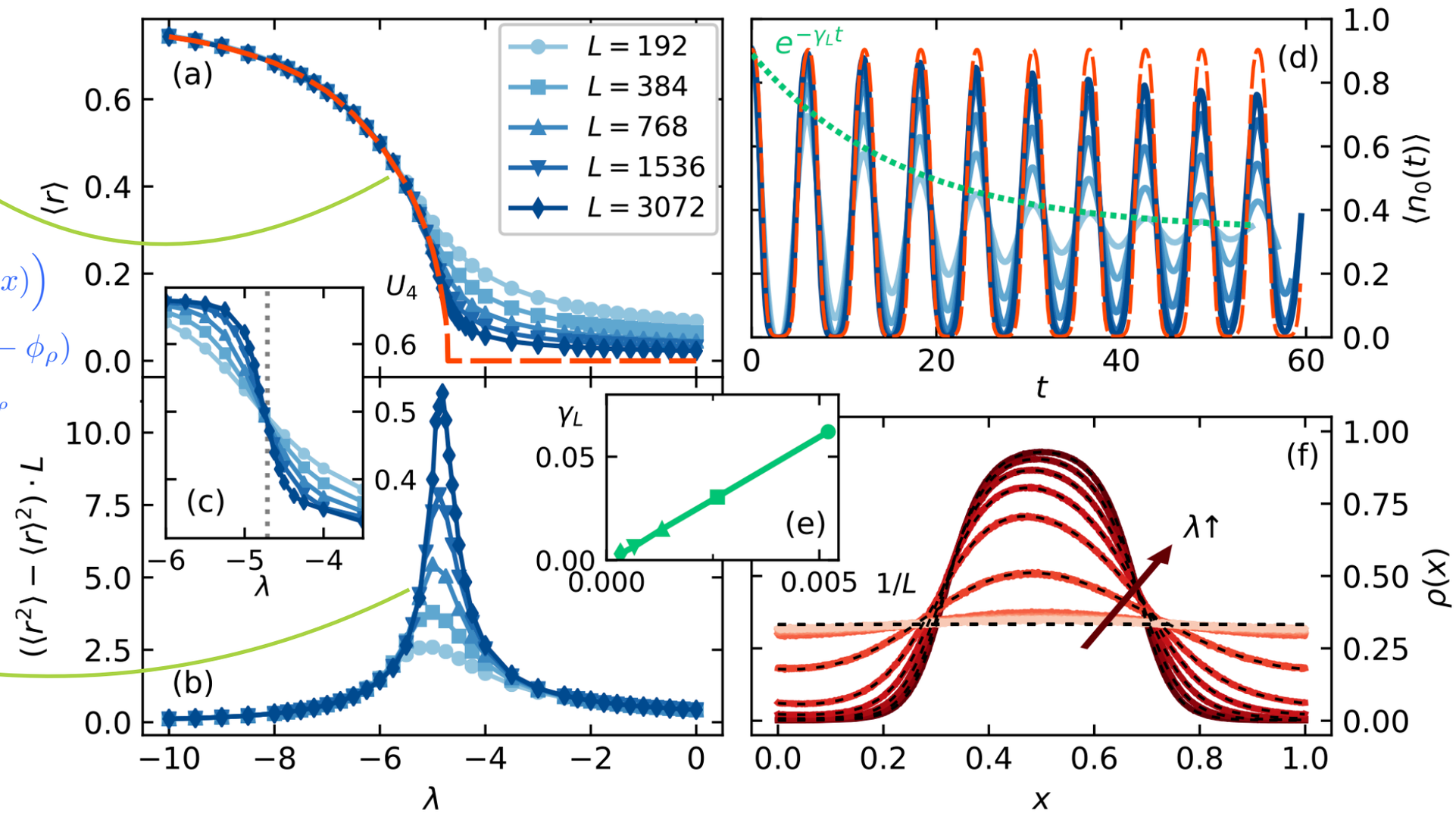
Divergent susceptibility as L grows



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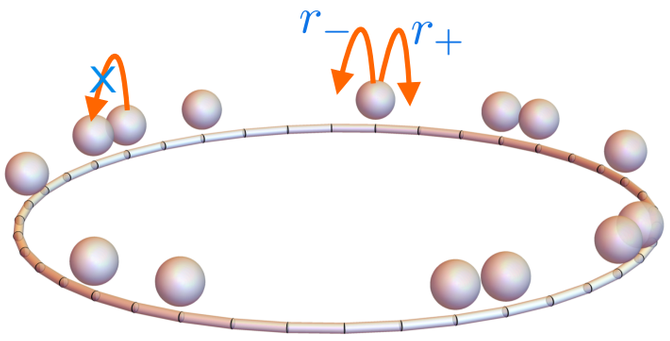

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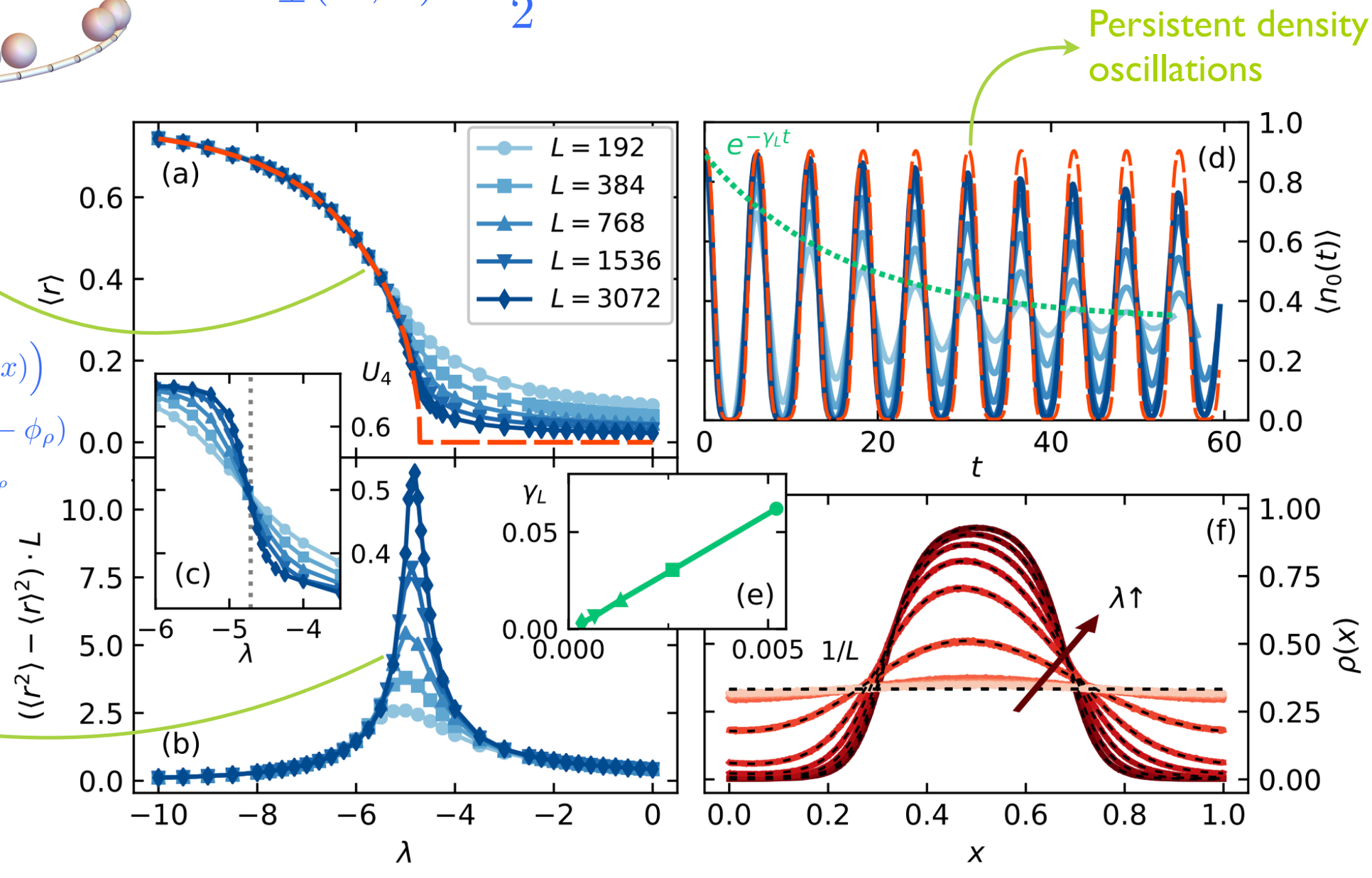
Divergent susceptibility as L grows

Persistent density oscillations



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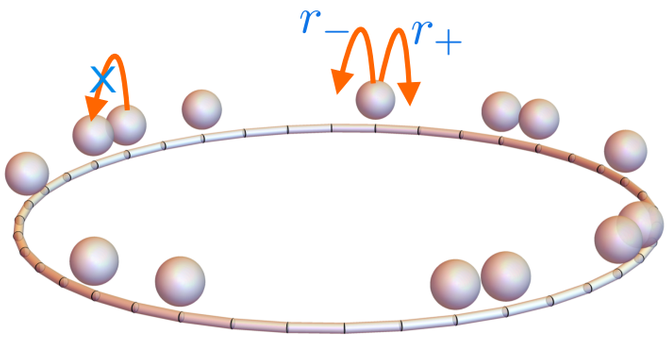

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Rigidity of long-range spatiotemporal order

Persistent density oscillations

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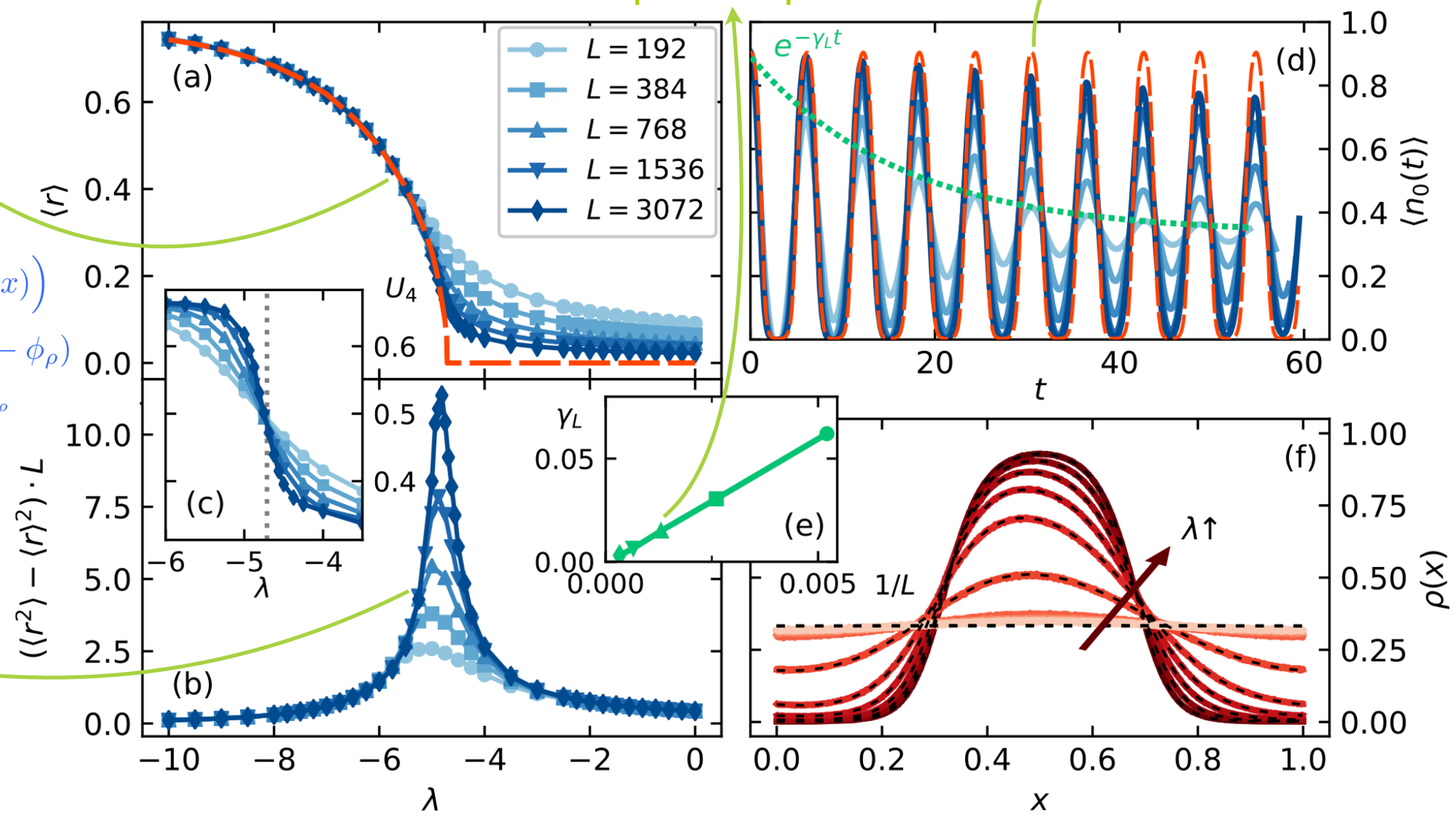


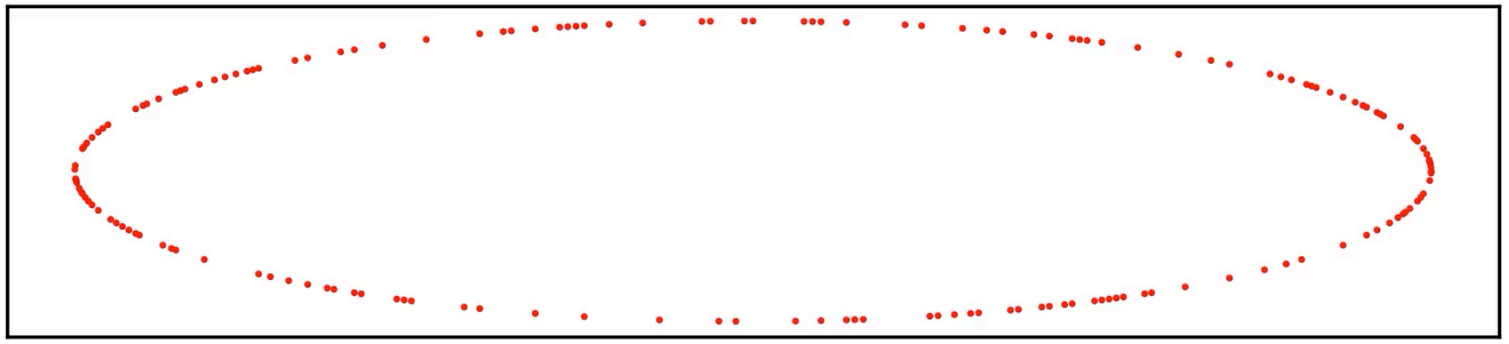
WHAT IF …?

$$E(\rho;\phi) = a + \sum_n b_n |z_\rho^{(n)}| \sin\left[\arg(z_\rho^{(n)}) - n\phi\right]$$

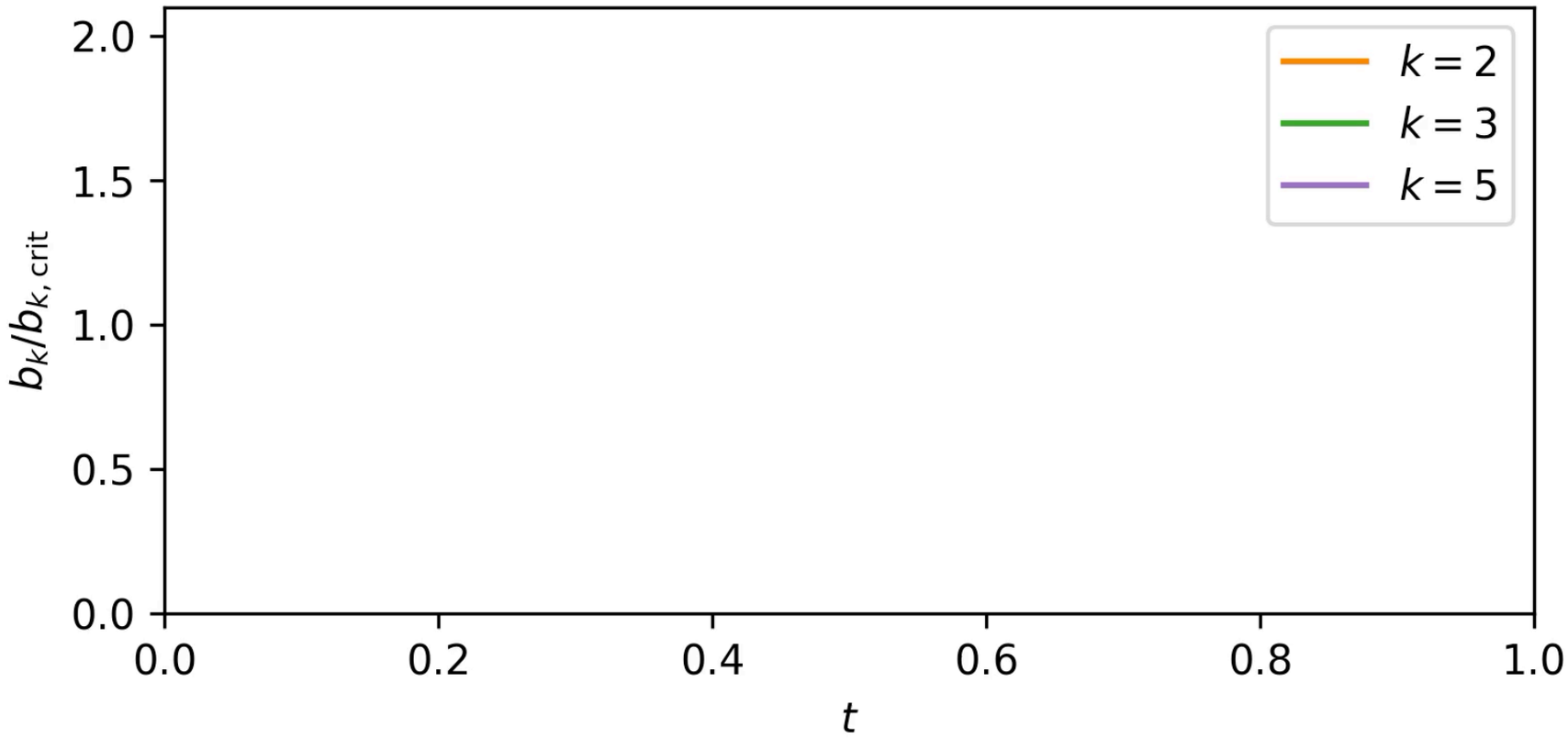
- **What happens if**, instead of exciting the first mode with the external packing field, **we excite any higher modes?**

$$z_\rho^{(n)} = \rho_0^{-1} \int_0^1 \rho(x) \mathrm{e}^{i2\pi n x}$$

- **Bingo!!**: we can engineer **multi-mode and controllable continuous time crystals**



$$b_{n,\mathrm{crit}} = \frac{2\pi n}{1-\rho_0}$$



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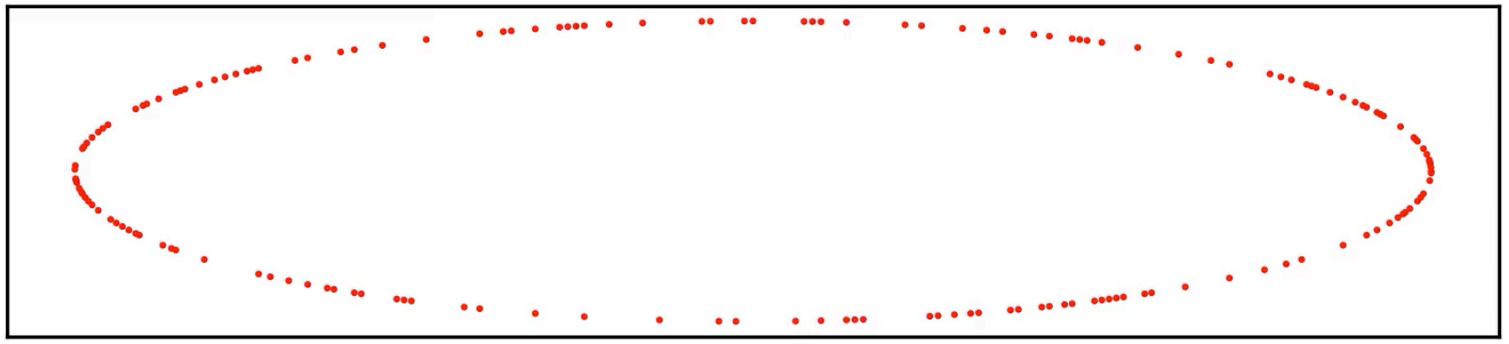

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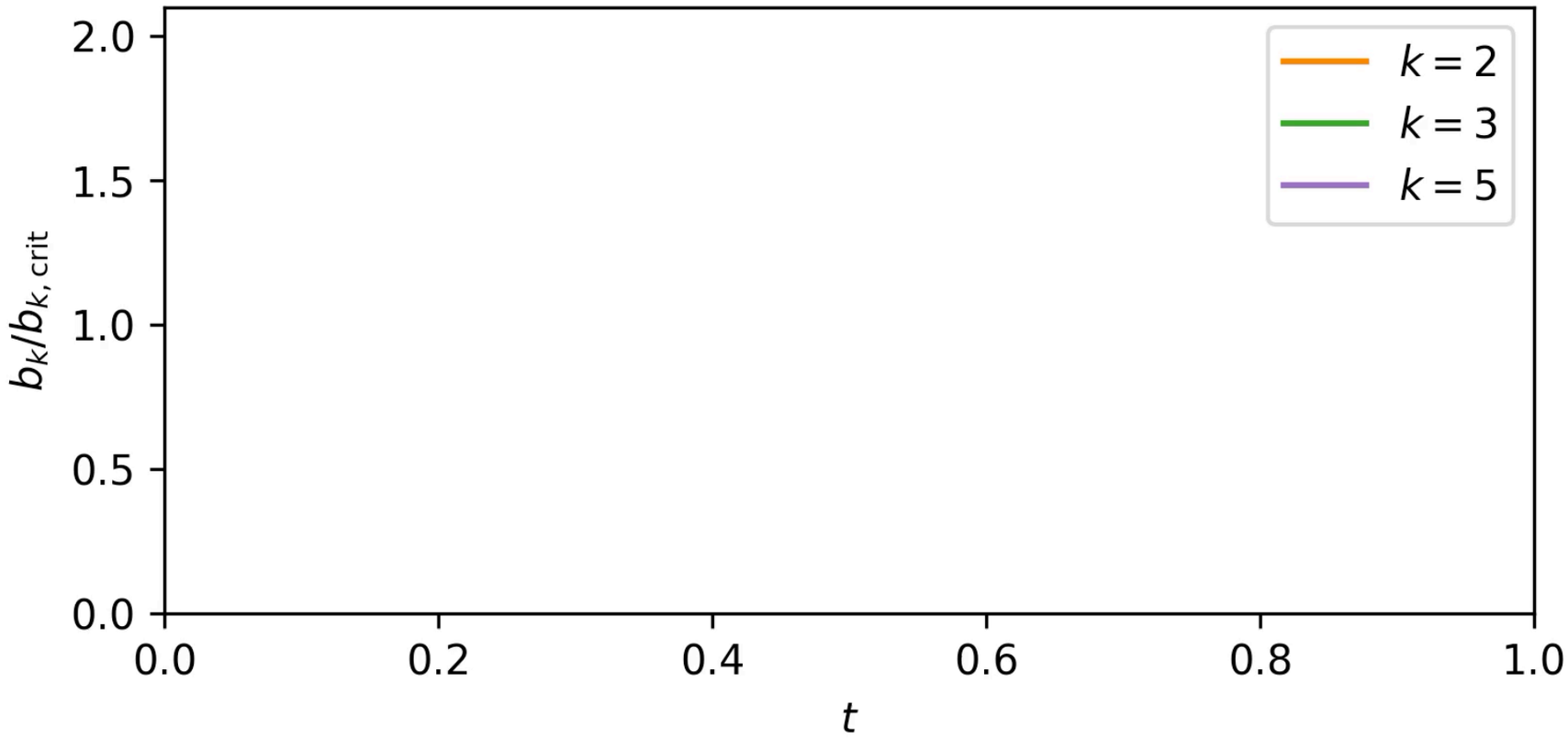


SUMMARY

- **Phase transitions forbidden in equilibrium** might be present at the level of **nonequilibrium fluctuations**

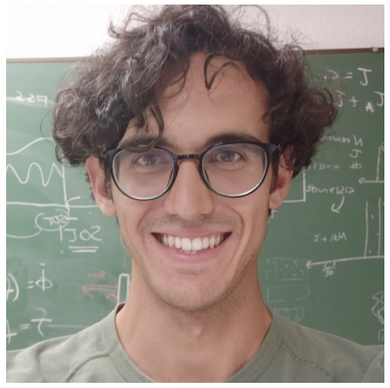
- A **time-crystal phase** has been identified for **unlikely small currents** in a simple diffusive system (WASEP)

- Despite being very rare, these phenomena **can be made typical** (observed in the stationary state)

- This allows us a systematic way of **'building'** these intriguing dynamical regimes from rare events (TCLG)

- Same approach could be exploited in **open quantum systems** (at last, $\hbar \neq 0$)

Carlos Pérez-Espigares
(Granada)



Rubén Hurtado
(Granada)

Federico Carollo
(Tübingen)

Thank you for your attention